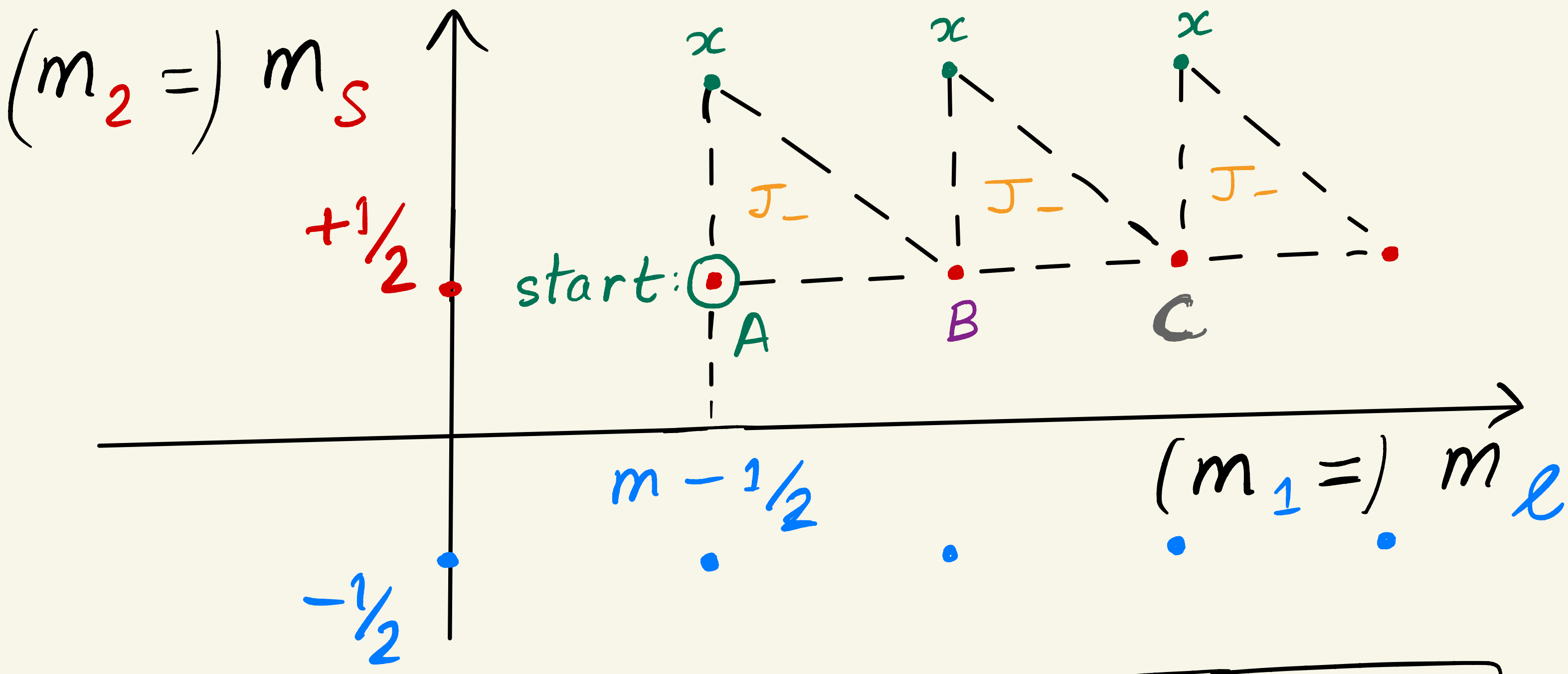
Lecture $\boxed{33}$, Nov. 16 (Mon.)

Outline for today (& Wed.)

- Calculation of CG coefficients for $l \oplus s\,(=1/2)$ as example of general strategy
- spin-angular part of wavefunction
- Connecting CG coefficients to rotation matrices
- ~~Schwinger's model: connecting SHO to angular momentum~~

Example of bootstraping to calculate CG coefficients for $l \oplus s\,(=1/2)$: $m_1 = m_l = -l \ldots$ to $+l$ & $m_2 = m_s = \pm 1/2$, with $j = l \pm 1/2$

- Only two rows in $m_1 = m_l$, $m_2 = m_s$ plane $\Rightarrow$ modify general strategy
- specialize (to begin with) to total $\boxed{j = l + 1/2}$
- Steps: (1). choose some m and $(m_2 =)\ \boxed{m_s = +1/2}$ $\boxed{\text{upper row}}$



(2). $\boxed{m_\ell}(= m_1) = m - m_1 \boxed{= m - 1/2}$

Parent/seed/starting point (A):

$(m - 1/2, +1/2)$

(3). As per (d) of general strategy, apply J_- recursion relation (lower triangular), with A on base, such that x is forbidden:

B [$m_s = +1/2$ (same as A), $m_\ell = m + 1/2$ (1 higher than A $\Rightarrow m+1$ vs. m for A]

gets related (only) to A (move right horizontally by 1)

$$\sqrt{(\ell+\tfrac{1}{2}+m+1)(\ell+\tfrac{1}{2}-m)}\,\overbrace{\langle \underbrace{m-\tfrac{1}{2}}_{(m_1=)\,m_\ell},\underbrace{\tfrac{1}{2}}_{(m_2=)\,m_s=+\frac{1}{2}} \,|\, \underbrace{\ell+\tfrac{1}{2}}_{j},\underbrace{m}_{\text{total}}\rangle}^{\text{CG of } A}$$

$$=\sqrt{(\ell+m+\tfrac{1}{2})(\ell-m+\tfrac{1}{2})}\,\underbrace{\langle m+\tfrac{1}{2},\tfrac{1}{2}\,|\,\ell+\tfrac{1}{2},m+1\rangle}_{\text{CG of } B}$$

$$\Rightarrow \underbrace{\langle m-\tfrac{1}{2},\tfrac{1}{2}\,|\,\ell+\tfrac{1}{2},m\rangle}_{\text{CG of } A}=\sqrt{\frac{\boxed{\ell+m+\tfrac{1}{2}}}{\boxed{\ell+m+\tfrac{3}{2}}}}\;\Big)^{-1}\quad\times$$

$$+(\ell+1)$$

$$\underbrace{\langle \overbrace{m+\tfrac{1}{2}},\tfrac{1}{2}\,|\,\ell+\tfrac{1}{2},m+1\rangle}_{\text{CG of } B}$$

(4). Keep applying J_- (relate B to C ...) sliding to right on $m_s=+1/2$ line:

$$\underbrace{\langle m-\tfrac{1}{2},\tfrac{1}{2}\,|\,\ell+\tfrac{1}{2},m\rangle}_{\text{CG of } A}=\sqrt{\frac{(\ell+m+\frac{1}{2})}{\cancel{(\ell+m+3/2)}}}\sqrt{\frac{\cancel{(\ell+m+3/2)}}{(\ell+m+5/2)}}\;\times$$

$$\underbrace{\langle m+\tfrac{3}{2}, \overset{\text{unchanged}}{\tfrac{1}{2} \,|\, l+\tfrac{1}{2}}, m+2\rangle}_{\text{CG of C}}$$

(5). Hopping to right along bases of J_- (lower) triangles

stops when we reach $m_l = +l$ (maximum allowed): further J_- gives 0

$$\Rightarrow \underbrace{\langle m-\tfrac{1}{2}, \tfrac{1}{2} \,|\, l+\tfrac{1}{2}, m\rangle}_{\text{CG of A}} = \sqrt{\frac{(l+m+\frac{1}{2})}{(2l+1)}} \times \text{CG of endpoint}$$

$\uparrow +(l+1)$

$$\langle \underset{\uparrow\ (\text{maximum})\, m_l}{l}, \tfrac{1}{2} \,|\, l+\tfrac{1}{2}, \underbrace{l+\tfrac{1}{2}}_{(\text{maximum})\ m}\rangle$$

(6). This "extreme" case: $m_l = l$ &

$m_s = 1/2$ ($\Rightarrow m = \ell + 1/2$) is not allowed for $\boxed{\text{other}}$ $j = (\ell - 1/2)$, since maximum m-value for latter is $(\ell - 1/2)$

$\Rightarrow |m_\ell = \ell, m_s = 1/2\rangle$ in old basis must be $= |\ell + 1/2, m + 1/2\rangle$ in new basis (up to phase: set to 1)

[For non-extreme m_ℓ/in general,

$|m_\ell, m_s = +1/2\rangle$ in old basis $=$

sum of $|j = \ell + 1/2, m = m_\ell + 1/2\rangle$

absent for extreme...

& $|j = \ell - 1/2, m = m_\ell + 1/2\rangle$]

$$\Rightarrow \langle \underset{m_\ell}{\ell}, \underset{m_s}{1/2} | \underbrace{\ell + 1/2}_{j}, \underbrace{\ell + 1/2}_{m} \rangle = 1$$

(CG of extreme right of $m_s = \frac{1}{2}$ line)

So, we get for CG of A,

$$\boxed{\langle m-\tfrac{1}{2}, \tfrac{1}{2} | l+\tfrac{1}{2}, m\rangle = \sqrt{\frac{(l+m+1/2)}{(2l+1)}}}$$

... but not done yet!

For $\boxed{\text{given}}$ m, there are three other CG coefficients:

$\langle \overbrace{m+1/2}^{m_l}, (-1/2) | l+1/2, m\rangle$ (on lower line of above figure)

& $\langle m \mp 1/2, \pm 1/2 | \underbrace{l - \tfrac{1}{2}}_{\text{other } j}, m\rangle$ (another figure)

– Instead of using recursion relations again, consider base kets involved:

$|l \pm 1/2, m\rangle$ in new basis &

$|\underbrace{m \mp 1/2}_{m_l}, \underbrace{\pm 1/2}_{m_s}\rangle$ $(m = m_l + m_s)$ in old basis

(for given m)

The above base kets form "closed system";

e.g., $|\ell+\tfrac{1}{2}, m\rangle$ (new basis) can not be expressed in terms of any other old base ket, since only above 2 ways to get m;

similarly $|\underbrace{m-\tfrac{1}{2}}_{m_1}, \overbrace{+\tfrac{1}{2}}^{m_2}\rangle$ (old basis) must be contained in $|\ell \pm \tfrac{1}{2}, m\rangle$ in new basis, since old base ket has $m_1+m_2=m$, matching new base kets

$\Rightarrow$ 2×2 orthogonal matrix to go between above pairs of base kets (its elements are CG coefficients):

$$|\ell-\tfrac{1}{2}, m\rangle = -\sin\alpha\, |m-\tfrac{1}{2}, \tfrac{1}{2}\rangle + \cos\alpha\, |m+\tfrac{1}{2}, -\tfrac{1}{2}\rangle$$

$$\langle m-\tfrac{1}{2}, +\tfrac{1}{2}|\ell+\tfrac{1}{2}, m\rangle = \langle m-\tfrac{1}{2}, +\tfrac{1}{2}|\, c_\alpha\, |m-\tfrac{1}{2}, +\tfrac{1}{2}\rangle + \langle m-\tfrac{1}{2}, +\tfrac{1}{2}|\, s_\alpha\, |m+\tfrac{1}{2}, -\tfrac{1}{2}\rangle$$

[Full CG-matrix is $(2j_1+1)\times(2j_2+1) =$ $2(2\ell+1)$-dimensional]
$(j_2 = 1/2;\ j_1 = \ell)$

Use above CG coefficient $\langle m-\tfrac{1}{2}, \tfrac{1}{2}|\ell+\tfrac{1}{2}, m\rangle$ on LHS and orthonormality on RHS gives

$$\boxed{c_\alpha = \sqrt{(\ell+m+1/2)/(2\ell+1)}}$$

sanity check: $(\ell+m+1/2) \leq (2\ell+1)$

$\Rightarrow s_\alpha = +\sqrt{(\ell - m + 1/2)/(2\ell+1)}$ (convention + bit of work: see below)

[In detail, $\sin\alpha = \langle \overbrace{m+1/2}^{m_1 (= m_\ell)}, \overbrace{-1/2}^{m_2 (= m_s)} | \ell + 1/2, m\rangle$, but all $j = \ell + 1/2$ states, such as $|\ell + \frac{1}{2}, m\rangle$, generated by J_- acting on $|\ell + 1/2 (= j), (m =) \ell + 1/2\rangle$, which is $|(m_1 = m_\ell =) \ell, (m_2 = m_s =) + \frac{1}{2}\rangle$ ... with J_- matrix elements (here in old m_1, m_2 basis) > 0 by choice ... all together, we get $\sin\alpha > 0$!]

So, "2^{nd}" CG coefficient for $\ell + 1/2$, i.e.,

$$\langle m + 1/2, -1/2 | \ell + \frac{1}{2}, m\rangle = s_\alpha \text{ etc...}$$

Aside: "total" (non-radial) wavefunction for above states (spin-angular functions)

$\uparrow$ 1/2 (spin) $\quad$ orbital (angular)

$$\mathcal{Y}_\ell^{j = \ell \pm 1/2, m \to m_\ell + m_s} \text{ (new basis)}$$

$$= \overbrace{\pm\sqrt{\frac{\ell \pm m + 1/2}{(2\ell+1)}}}^{\cos\alpha \text{ or } -\sin\alpha} \underbrace{Y_\ell^{\underbrace{m-1/2}_{m_\ell}}(\theta, \phi)}_{\text{orbital part}} \underbrace{\chi_+ \left(= \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)}_{\substack{\text{spin part} \\ m_s = +1/2}}$$ (see above)

$$+ \overbrace{\sqrt{\frac{\ell \mp m + 1/2}{2\ell+1}}}^{\sin\alpha \text{ or } \cos\alpha} Y_\ell^{m+1/2}(\theta, \phi) \underbrace{\chi_-}_{\begin{pmatrix} 0 \\ 1 \end{pmatrix}: m_s = -\frac{1}{2}}$$

simultaneous eigenkets of

$|\bar{L}|^2, |\bar{S}|^2, |\bar{J}|^2$ & J_z, but **not** of L_z, S_z (separately)

Also of $\bar{L}\cdot\bar{S} = \frac{1}{2}\left(|\bar{J}|^2 - |\bar{L}|^2 - |\bar{S}|^2\right)$

(hence, relevant for spin-orbit coupling:

these are (new) energy wavefunctions :

again eigenstates of $\bar{L}\cdot\bar{S}$ (interaction) in H

& $|\bar{L}|^2$ contained in kinetic energy

$$\bar{L}\cdot\bar{S} \text{ eigenvalue} = \frac{\hbar^2}{2}\left[j(j+1) - \ell(\ell+1) - \frac{3}{4}\right]$$

$$= \begin{cases} \ell\,\hbar^2/2 & \text{for } j = \ell + 1/2 \\ -\frac{\hbar^2}{2}(\ell+1) & \text{for } j = \ell - 1/2 \end{cases}$$

($\Rightarrow$ degeneracy in ℓ & s lifted)