Lecture $\boxed{32}$, Nov. 13 (Fri.): part (II)

Outline: continue properties of CG coefficients with goal of computing CG coefficients (which determine transition amplitudes between initial & final states which are "combinations" of angular momenta (spins and/or orbital motion)

(5). $\boxed{\text{Recursion relation}}$ for CG coefficients

- connects CG coefficients with different indices [like with solving DE by series ansatz: here, used to compute CG coefficients (next topic)]

- Recall definition of CG coefficients:

$$\underbrace{|j_1 j_2; j m\rangle}_{\substack{\text{basis (B)} \\ \text{(new)}}} = \sum_{m_1} \sum_{m_2} \underbrace{|j_1 j_2; m_1 m_2\rangle}_{\text{basis (A) (old)}} \times (\text{CG coefficient})$$

$$\boxed{\langle j_1 j_2; m_1 m_2 | j_1 j_2; j, m\rangle}$$

– Consider fixed j_1, j_2, so for simplicity "drop" $j_1 j_2$ label : $\langle \underbrace{m_1 m_2}_{\text{row index}} | \underbrace{j\, m}_{\text{column index}} \rangle$

– with $j_{min} = |j_1 - j_2| \leq j \leq j_{max} = (j_1 + j_2)$

– Schematically, recursion relation :

$$\langle m_1 m_2 | j\ m \pm 1 \rangle \sim \langle m_1 \mp 1\ m_2 | j\, m \rangle + \langle m_1 m_2 \mp 1 | j, m \rangle$$

Features

(i). all 3 CG coefficients have same j (part of "column" index)

(ii) Rest of column index (m) : same for 2 RHS terms, but LHS is different by 1 ("net" column index different for LHS vs. full RHS)

(iii) "Row" index $(m_1\, m_2)$: same m_2 for LHS & 1st RHS, but 2nd RHS different ; m_1 same LHS & 2nd RHS, but different for 1st RHS

⇒ "net" row index different for all 3 terms

Derivation

(1). $$J_{\pm} |j m\rangle = (J_{1\pm} + J_{2\pm}) \sum_{m_1' m_2'} |\overset{j_1}{m_1'}, \overset{j_2}{m_2'}\rangle \times$$

$$\langle m_1' m_2' | j m\rangle \Big\}$$ CG coefficient (number)

(2) Use general result:

$$J_{\pm} |j\, m\rangle = c^{\pm}_{jm} |j\ m \pm 1\rangle ,$$

where $$c^{\pm}_{jm} = \sqrt{(j \mp m)(j \pm m + 1)}$$

(Just do J_+ ; J_- is analogous)

$$c^{+}_{jm} \overbrace{\langle m_1, m_2}^{\text{"fixed" (cf. } m_1' m_2')} | j\ m+1\rangle = \langle m_1' m_2' | j m\rangle \times$$

$$\sum_{m_1' m_2'} \left(c_{j_1 m_1'} \langle m_1 m_2 | m_1'+1\ m_2'\rangle + c_{j_2 m_2'} \langle m_1 m_2 | m_1'\ m_2'+1\rangle \right)$$

(3) Use orthonormality of $|m\ m'\rangle$'s:

$$\langle m_1\ m_2 | m_1'\ m_2'\rangle = \delta_{m_1 m_1'}\delta_{m_2 m_2'}$$

$\Rightarrow$ 1st term on above RHS:

$\langle m_1\ m_2 | m_1'+1\ m_2'\rangle$ sets

$m_1' = m_1 - 1$ & $m_2' = m_2$

Similarly, in 2nd term: $m_1' = m_1$

& $m_2' = m_2 - 1$

So, recursion relation

$$\boxed{\begin{aligned} c^{+}_{jm}\langle m_1 m_2 | j\ m+1\rangle &= c^{+}_{j_1\, m_1-1}\langle m_1-1\ m_2 | jm\rangle \\ &\quad + c^{+}_{j_2\, m_2-1}\langle m_1\ m_2-1 | j\ m\rangle \end{aligned}}$$

Similarly, using J_- gives

$$\boxed{\begin{aligned} c^-_{jm}\langle m_1 m_2 | j\, m-1\rangle &= c^-_{j_1\, m_1+1}\langle m_1+1\; m_2 | jm\rangle \\ &+ c^-_{j_2\, m_2+1}\langle m_1\; m_2+1 | jm\rangle \end{aligned}}$$

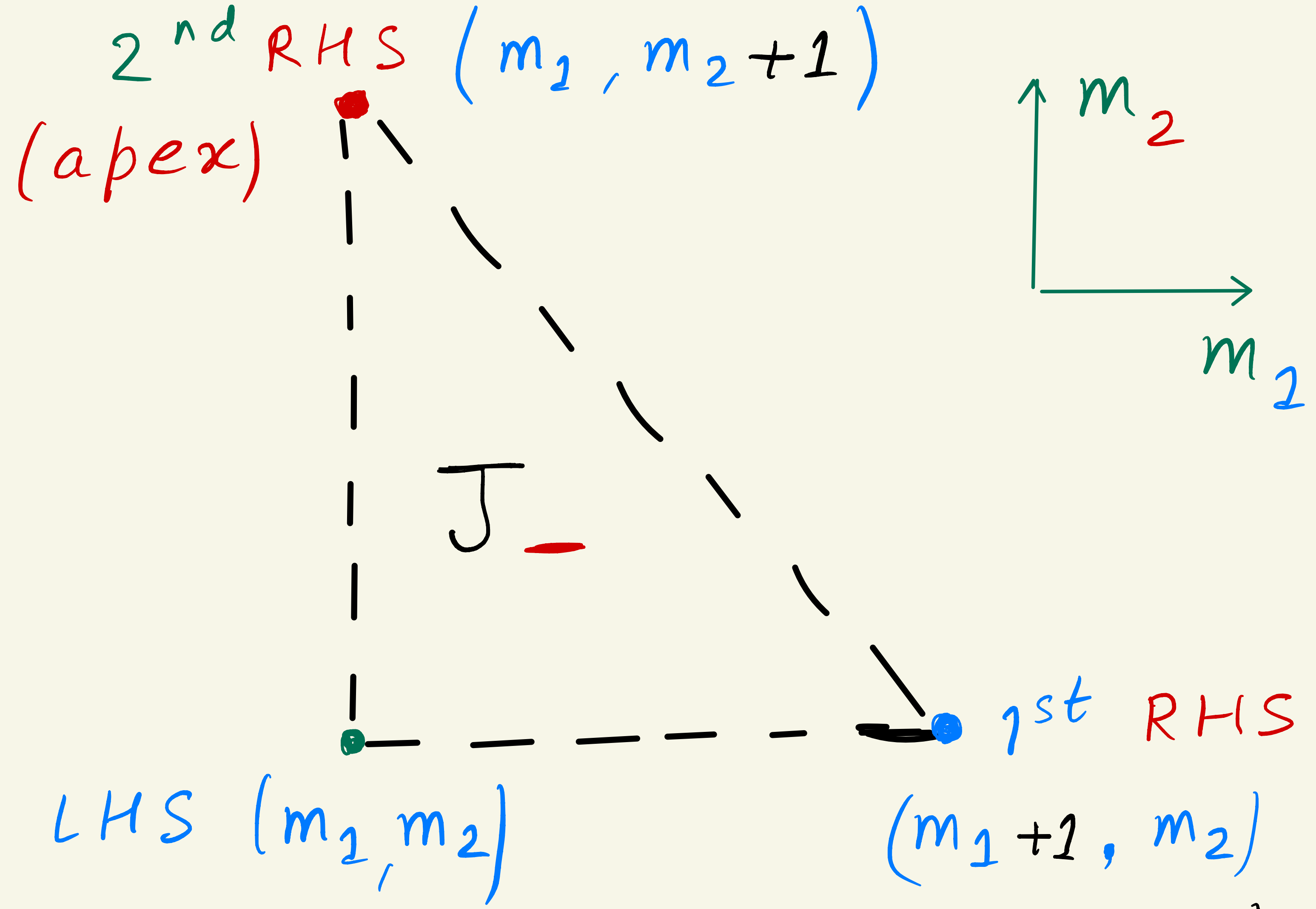
Note: for CG coefficients to be non-vanishing, we need as per property (1):

$$\boxed{m_1 + m_2 = m \pm 1}$$ (depending on 1st/2nd)

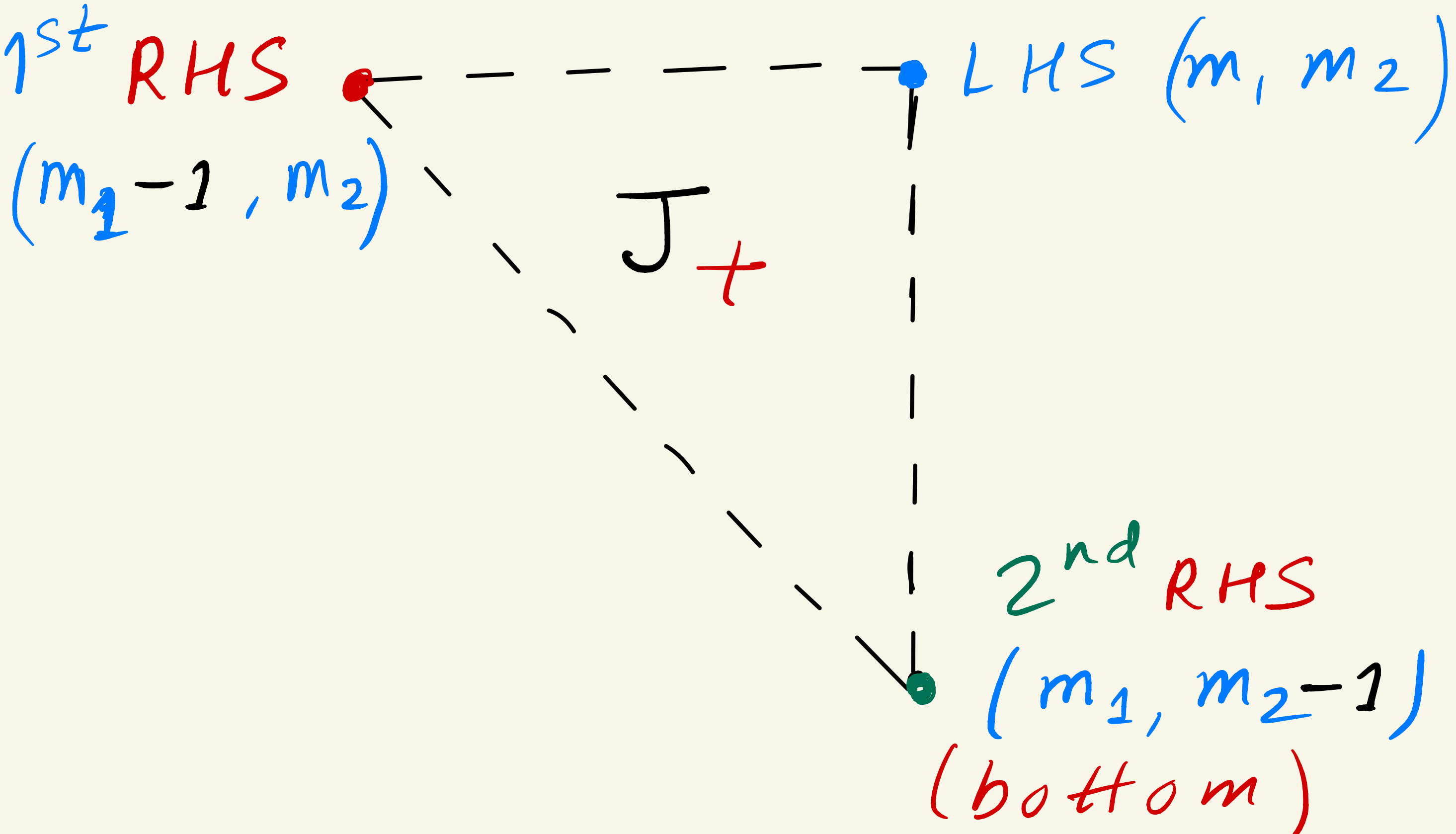
but this is not explicitly applied above

- Display recursion relation in m_1, m_2-plane: J_- relates CG at (m_1, m_2) to (m_1, m_2+1) & (m_1+1, m_2) [all 3 are "points" in plane]

$\Rightarrow$ a triangle



Similarly, J_+ ("upper triangular")



Finally, computing CG coefficients

Basic strategy :

(a). choose one [see (d)] CG coefficient as "parent/seed" (unknown at this stage : to be fixed in last step)

(b). Use recursion relations to express other CG coefficients in terms of seed (idea: bootstrap)

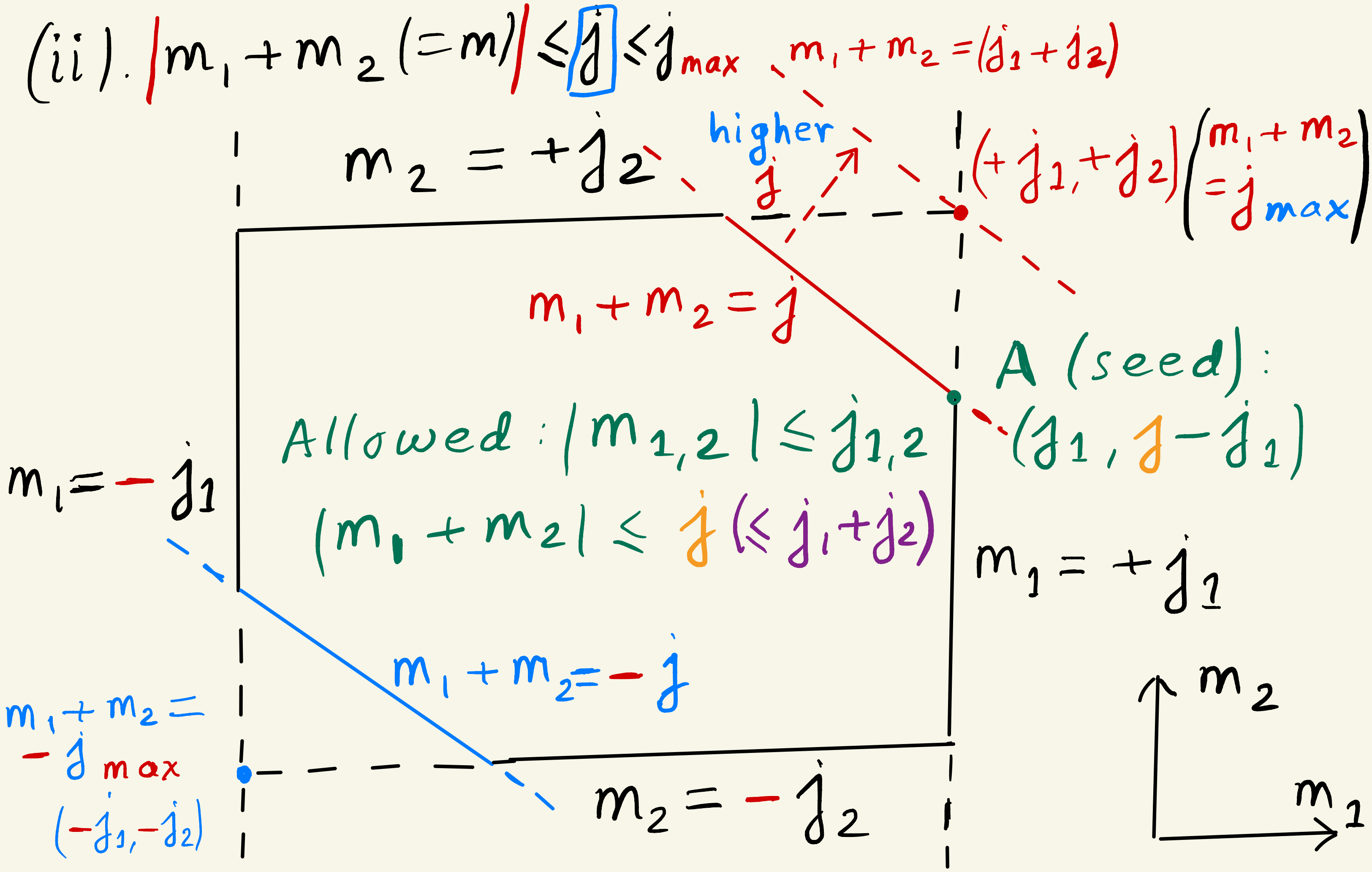
(c). Fix seed by normalization condition [property (4)], which involves sum over (squares of) CG coefficients (now all written in terms of seed)

(d). However, recursion relation (in general) involves 3 CG coefficients (so "complicated" to express other CG coefficients in terms of one seed) $\boxed{?!}$

No worries! For seeding, choose a special case where $\boxed{\text{one}}$ CG coefficient vanishes due to raising/lowering of highest/lowest allowed m ⇒ bootstrap will work!

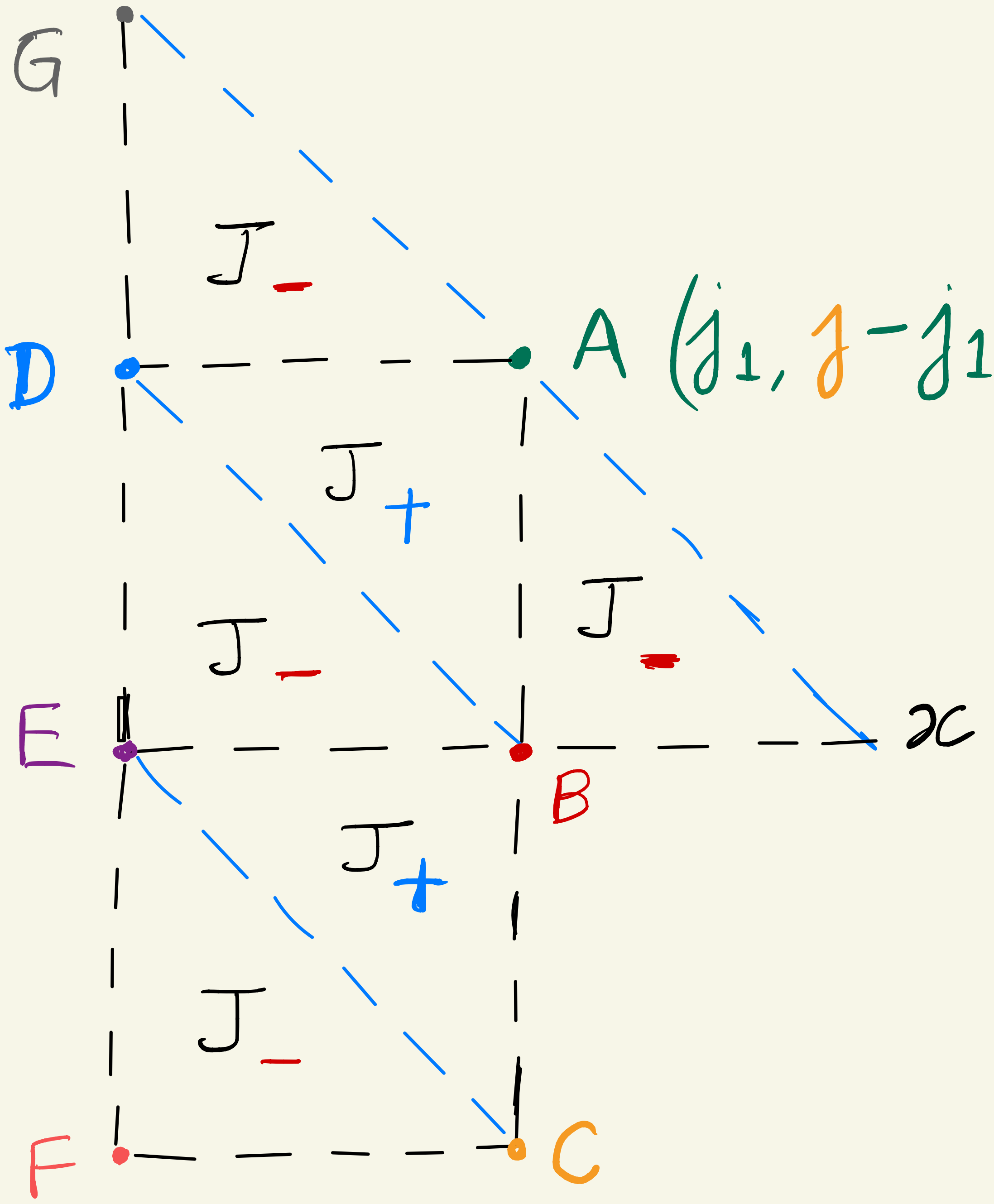
Let's do it! Consider (m_1, m_2) plane for fixed $j_{1,2}$ and some choice of j in range

First, (I). chart "playground" (allowed $m_{1,2}$) based on (i). $|m_{1,2}| \leq j_{1,2}$ & (ii). $|m_1 + m_2 (= m)| \leq \boxed{j} \leq j_{max}$



Play ball :

(II). Start at A (upper right corner), will implement strategy (d) :



(III). Use J_- (lower triangular) :

$\langle m_1 m_2 | j\ m-1 \rangle \sim \langle m_1+1\ m_2 | j\ m \rangle + \langle m_1\ m_2+1 | j\ m \rangle$

with $A(\underset{m_1}{j_1}\ \ j-j_1)$ being 2nd RHS (apex of ▷)

⇒ 1st RHS has $m_1+1 = j_1+1$, forbidden (point $\boxed{x}$ outside of playground) ⇒

get rid of 1 CG in recursion relation

B (as LHS) connected to A only:

CG of B expressed in terms of CG of A

(IV). To get to D, use J_+ (upper triangular) on ABD ⇒ CG of D in terms of CG's A & B

(all 3 CG's non-vanishing, but Ok!

⇒ given (III) [relates CG of A & B], CG of D from CG of A ... (continue)

CG of E from Δ DBE ...

(V). Similarly, go up in m_2 :

Δ GAD gives CG of G in terms of CG of A [via CG of D from (IV)]

... every CG of playground gotten in terms of CG of A (only $\boxed{1}$ unknown)

... fix CG of A (hence rest) from normalization condition (sum of squares of CG's)

Example : add $\bar{L}$ and $\bar{S}$ of single spin-$1/2$ particle (fine structure of one-electron atom) so that $j_1 = \ell$; $m_1 = m_\ell \left[= -\ell, -\ell+1 \ldots 0 \ldots \ell-1, \ell\right]$ & $j_2 = s = 1/2$; $m_2 = m_s \left[= \pm 1/2\right]$ ⇒ allowed $j = \left(\ell \pm 1/2\right)$ for $\ell > 0$ $\left(j = 1/2 \text{ for } \ell = 0\right)$,

e.g., $\ell = 1$ (p-wave) gives $P_{j=3/2,\,1/2}$