Lecture $\boxed{31}$, Nov. 11 (Wed.)

(continued from lecture 30)

Clebsch-Gordon (CG) coefficients (in general):

unitary transformation (U) relating bases (A) & (B)

Recall general change of basis:

$$\underbrace{|b^{(\ell)}\rangle}_{\text{eigenkets of } B \text{ (new basis)}} = \underbrace{\sum_k |a^{(k)}\rangle\langle a^{(k)}|}_{1}\, b^{(\ell)}\rangle$$

U-matrix elements

- Here, $b^{(\ell)} \longrightarrow |j_1 j_2; j m\rangle$, eigenkets of $\boxed{|\bar{J}|^2, J_z, |\bar{J}_{1,2}|^2}$

  $a^{(k)} \longrightarrow |j_1 j_2; m_1 m_2\rangle$, eigenkets of $\boxed{|\bar{J}_{1,2}|^2, J_{1,2z}}$

U-matrix elements $\overset{\text{row} \rightarrow}{\langle \underbrace{a^{(k)}}_{\text{row}}} \overset{\rightarrow \text{column}}{|\underbrace{b^{(\ell)}}_{\text{new}}\rangle} \longrightarrow \boxed{(\boxed{CG}\text{ coefficients})}$

old basis (row) $\rightarrow$ $\boxed{\langle j_1 j_2; m_1 m_2 | j_1 j_2; j m\rangle}$ $\rightarrow$ new basis (column)

$\boxed{\text{Properties}}$ of CG coefficients (useful for computing them)

(1). Non-vanishing for $\underset{\text{(new)}}{m} = m_1 + m_2$, since

$$\langle j_1 j_2; m_1 m_2| \left[J_z - (J_{1z} + J_{2z})\right] |j_1 j_2; j m\rangle = 0$$

$$= \langle j_1 j_2; m_1 m_2 | j_1 j_2; j m\rangle \left[m - (m_1 + m_2)\right]$$

⇒

(2). Non-vanishing only for $j_{min} \le j \le j_{max}$:

$j_{max} = (j_1 + j_2)$, since maximum $m \, (= m_1 + m_2)$
$= j_1 + j_2$

$j_{min} = |j_1 - j_2|$ (proof in Sakurai appendix C)

(sanity) check: dimensionality of spaces agree in 2 bases (fixed $j_{1,2}$)

space of basis (A) $\{|j_1 \, j_2 ; m_1 m_2\rangle\}$ has dimensionality $N_A = \begin{pmatrix}\text{number of} \\ \text{allowed } m_1 \\ \text{values}\end{pmatrix} \times \begin{pmatrix} \qquad m_2 \end{pmatrix}$

$= (2j_1+1) \times (2j_2+1)$

dimensionality of space of basis (B), N_B

$$= \sum_j \begin{array}{l}\text{number of} \\ \text{allowed } m \\ \text{values for given } j\end{array} = \sum_{|j_1-j_2|}^{(j_1+j_2)} (2j+1)$$

$$= 2 \sum_{|j_1-j_2|}^{(j_1+j_2)} j + \sum_{|j_1-j_2|}^{(j_1+j_2)} 1$$

$$= 2\left[\sum_{1}^{(j_1+j_2)} j - \sum_{1}^{|j_1-j_2|-1} j\right] + \left[(j_1+j_2) - |j_1-j_2| + 1\right]$$

$$\left[\text{Use } \sum_{1}^{n} m = n(n+1)/2\right]$$

assume (no loss of generality) $j_1 > j_2$

$$= 2\left[\frac{(j_1+j_2)(j_1+j_2+1)}{2} - \frac{(j_1-j_2-1)(j_1-j_2)}{2}\right] + (2j_2+1)$$

$$= \left[(j_1+j_2)^2 + (j_1+j_2) - (j_1-j_2)^2 + (j_1-j_2)\right] + (2j_2+1)$$

$$= 4j_1j_2 + 2j_1 + 2j_2 + 1$$

$$= (2j_1+1)(2j_2+1)$$

... as above

(3). CG coefficients chosen to be real

CG matrix (unitary) is orthogonal

CG-"inverse" = CG-transpose

matrix element of CG-inverse

$$= \boxed{\underbrace{\langle j_1 j_2; j, m}_{\uparrow\ \text{row index (new basis)}} \,|\, \underbrace{j_1 j_2; m_1, m_2\rangle}_{\uparrow\ \text{column index (old basis)}}}$$

= matrix element of CG-transpose, with same row & column

= matrix element of CG, with row & column exchanged

$$\boxed{= \langle j_1 j_2 ; m_1 m_2 | j_1 j_2 ; j\, m \rangle}$$

[as expected: in general, $\langle \alpha | \beta \rangle = \langle \beta | \alpha \rangle^*$

Here, $\langle \alpha | \beta \rangle$, with $|\beta\rangle = |j_1\, j_2 ; m_1 m_2 \rangle$ & $|\alpha\rangle = |j_1\, j_2 ; j\, m\rangle$, is $\boxed{\text{real}}$ $\Rightarrow \langle \alpha | \beta \rangle = \langle \beta | \alpha \rangle$]

(4). Sum of product of CG coefficients

← "drop" $j_{1,2}$ label

$$\langle j_1\, j_2 ; m_1 m_2 | \underbrace{\left(\sum_{jm} |j m\rangle \langle j m| \right)} |j_1\, j_2 ; m_1' m_2' \rangle$$

$\Uparrow$

(orthonormality of $\langle j_1 j_2 ; m_1 m_2 \rangle$

$$= \boxed{\delta_{m_1 m_1'} \, \delta_{m_2 m_2'}}$$

$$= \sum_j \sum_m \langle j_1 j_2 ; m_1 m_2 | j_1 j_2 ; j\, m \rangle \times \langle j_1 j_2 ; m_1' m_2' | j_1 j_2 ; j\, m \rangle^*$$

↪ real

$$= \boxed{\sum_j \sum_m \langle j_1 j_2 ; m_1' m_2' | j_1 j_2 ; j\, m \rangle \times \langle j_1 j_2 ; m_1 m_2 | j_1 j_2 ; j m \rangle}$$

[can get it from CG orthogonality also:

$$(CG)(CG)^T = 1 \Rightarrow \boxed{\sum_c} (CG)_{r'c} \underbrace{\left[(CG)^T\right]_{cr}}_{(CG)_{rc}\,\}\text{"switched"}} = \boxed{\delta_{r'r}}$$

where "c" is column index (new basis): $|j_1 j_2; j m\rangle$

& "r, r′" are row indices (old basis): $|j_1 j_2; m_1 m_2$ or $m_1' m_2'\rangle$ so that we get

$$\sum_{jm} \langle m_1' m_2' | j m\rangle \langle m_1 m_2 | j m\rangle = \delta_{m_1' m_1}\, \delta_{m_2' m_2}$$

$\longrightarrow$ "drop" $\boxed{j_1 j_2}$ label

Similarly,

$$\boxed{\begin{aligned} \sum_{m_1, m_2} \langle j_1 j_2; m_1 m_2 | j_1 j_2; j m\rangle \times \\ \langle j_1 j_2; m_1 m_2 | j_1 j_2; j' m'\rangle \\ = \delta_{jj'}\, \delta_{mm'} \end{aligned}}$$

Special case: set $j = j'$, $m = m'$

$$\sum_{m_1, m_2} \left|\langle j_1 j_2; m_1 m_2 | j_1 j_2; j m\rangle\right|^2 = 1$$

(gives normalization for CG

(of course, need $m = m_1 + m_2$)