Lecture $\boxed{30}$, Nov. 9 (Mon.)

Outline for today (& next few lectures)

- $\boxed{\text{Addition}}$ of angular momenta
  - warm-ups: adding $\bar{L}$ & $\bar{S}$ for same particle & $\bar{S}$ for 2 particles
  - key ideas: (1) eigenvalues/states of net angular momentum and (2) change of basis from eigenkets of seperate angular momenta to those of net: Clebsch-Gordon (CG) coefficients
  - general method for addition of angular momenta
- Motivation: (a). illustrates basis change + (b). applications: atomic spectroscopy (spin-orbit interaction); selection rules (angular momentum conservation) in collisions ...

Example 1: adding spin, $\bar{S}$ (intrinsic angular momentum) to orbital angular momentum, $\bar{L}$ of same particle, as in spin-orbit coupling for one-electron atom

[recall: started course with spin-$1/2$ system (only those two d.o.f.) ... later, orbital angular momentum from spatial motion, but without including spin]

$\Rightarrow$ base ket for spin-$1/2$: "direct product" space of ∞-dimensional space of position eigenkets $\{|\bar{x}'\rangle\}$ and 2-d spin space ($|\pm\rangle$ base kets):

$$|\bar{x}', \pm\rangle = |\bar{x}'\rangle \otimes |\pm\rangle$$

operator in $|\bar{x}'\rangle$ space, e.g., $\bar{L}$ commutes with operator acting on $|\pm\rangle$, e.g., $\bar{S}$ (mutually independent vector spaces)

— Total angular momentum:

$$\bar{J} \equiv \bar{L} \otimes \mathbb{1} + \mathbb{1} \otimes \bar{S} \equiv \text{"} \bar{L} + \bar{S} \text{"}$$

in spin-space $\quad\quad$ in $|\bar{x}'\rangle$ space

– Check: $[J_i, J_j] = i\varepsilon_{ijk} J_k$;

$\boxed{[|\bar{J}|^2, J_z] = 0}$ ($\Rightarrow$ eigenkets of $|\bar{J}|^2$ & J_z)

$\Rightarrow$ $\bar{J}$ is also angular momentum

– Rotation operator for entire system:

$$\mathcal{D}(R) = \exp\left(\frac{-i\,\bar{J}\cdot\hat{n}\phi}{\hbar}\right) = \exp\left(\frac{-i\bar{L}\cdot\hat{n}\phi}{\hbar}\right) \otimes \exp\left(\frac{-i\bar{S}\cdot\hat{n}\phi}{\hbar}\right)$$

$$= \mathcal{D}_{orbital}(R) \otimes \mathcal{D}_{spin}(R)$$

same

– Wavefunction for particle with spin-½: $\langle \bar{x}', \pm | \alpha \rangle = \psi_\pm(\bar{x}')$

arrange in column: $[\psi_+(\bar{x}')\ \ \psi_-(\bar{x}')]^T$

($|\psi_\pm(x')|^2$ is probability for particle to be located at $\bar{x}'$ with spin up/down)

– Use $|n, l, m\rangle$ as base kets vs. $|\bar{x}'\rangle$:

$|\bar{L}|^2$ eigenvalue $l(l+1)\hbar^2$ and $m\hbar$ for L_z

$|\pm\rangle$ are already eigenkets of $|\bar{S}|^2$ & S_z with eigenvalues $\frac{1}{2}\left(\frac{1}{2}+1\right)\hbar^2 = \frac{3}{4}\hbar^2$ & $\pm\hbar/2$

obvious/old

So, basis (A) [eigenkets of $\boxed{|\bar{L}|^2, L_z, |\bar{S}|^2, S_z}$]:

$$\boxed{|l, s; m_l, m_s\rangle}$$

m_l: $-l$ to $+l$; m_s: $-s$ to $+s$

- Check: $|\bar{J}|^2 = |\bar{L}|^2 + |\bar{S}|^2 + 2L_zS_z + L_+S_- + L_-S_+$ ($L_{x,y}$)

$\Rightarrow [|\bar{J}|^2, L_z] = -[|\bar{J}|^2, S_z] \neq 0$ (due to $[L_+S_-, S_z]$ ...)

due to $[|\bar{J}|^2, J_z(=L_z+S_z)] = 0$

$\Rightarrow |l, s; m_l, m_s\rangle$ is **not** eigenket of $|\bar{J}|^2$, but is eigenket of $J_z(=L_z+S_z)$ (since it is of S_z & L_z separately)

New Basis (B): $[|\bar{J}|^2, |\bar{L}|^2] = 0 = [|\bar{J}|^2, |\bar{S}|^2]$

$\Rightarrow$ eigenkets of $\boxed{|\bar{J}|^2, J_z, |\bar{L}|^2, |\bar{S}|^2}$, but **not** of L_z, S_z separately

$$\boxed{|l, s; j, m\rangle}$$

l: gives $|\bar{L}|^2$ eigenvalue; s: $|\bar{S}|^2$ ...

j: $|\bar{J}|^2$ eigenvalue

m: $J_z(=L_z+S_z)$ eigenvalue

Example 2 : add spin-1/2 of two different particles : e.g. hydrogen atom : spins of proton & electron (hyperfine splitting : $H \propto$ (B due to proton spin) . spin of electron $\propto \bar{S}_p \cdot \bar{S}_e \ldots$

$$\bar{S} \equiv \bar{S}_1 \otimes \mathbb{1} + \mathbb{1} \otimes \bar{S}_2 \equiv \bar{S}_1 + \bar{S}_2$$

$$\left([S_{1x}, S_{2y}] = 0 \; ; \; [S_{1x}, S_{1y}] = i\hbar S_{1z} \; ; \right.$$

$$[S_x, S_y] = i\hbar S_z \; ; \; [|\bar{S}|^2, S_z] = 0 \; ; \; [|\bar{S}|^2, |S_1|^2] = 0 \; ;$$

$[|\bar{S}|^2, S_{1z}] \neq 0$, since $|\bar{S}|^2 \ni S_{1x/y} S_2 \ldots$

Basis (A) [eigenkets of $|\bar{S}_1|^2, S_{1z}, |\bar{S}_2|^2, S_{2z}$] : can not "add $|\bar{S}|^2$: $[|\bar{S}|^2, S_{1z}] \neq 0$

$\boxed{|s_1, s_2 ; m_1, m_2\rangle}$: $\boxed{s_1 = s_2 = 1/2 \; ; \; m_1 = \pm 1/2 \; ; \; m_2 = \pm \frac{1}{2} \text{ (here)}}$

s_1: $|\bar{S}_1|^2$ eigenvalue; m_1: S_{1z} eigenvalue

... denoted simply by $\boxed{|m_1 \; m_2\rangle}$:

$|++\rangle, |+-\rangle, |-+\rangle, |--\rangle$ (4 base kets)

Basis (B) [eigenkets of $|\bar{S}|^2, S_z, |\bar{S}_1|^2, |\bar{S}_2|^2$: can not "add" S_{1z}, S_{2z} : $[|\bar{S}|^2, S_{1z}] \neq 0$] :

$\boxed{|s_1, s_2 ; s, m\rangle}$ or simply $\boxed{|s, m\rangle}$,

s, m are eigenvalues of $|\vec{S}|^2$ & S_z

Note: two bases have same number of states ("simply re-writing" to go from one to other)

(I). $\boxed{\text{Find}}$ allowed values of s ($m = -s$ to $+s$)

$|m_1, m_2\rangle$ of basis (A) eigenket of S_z ($= S_{1z} + S_{2z}$) with eigenvalue $m = m_1 + m_2$ $\Rightarrow$ m is integer (only)

$\Rightarrow$ s is integer (only) $= 0, 1, 2 \ldots$

Maximum (minimum) $m = +1/-1$ $\Rightarrow$ (based on $m_{1,2}$ in old basis, but "independent" of basis)

maximum (and achieved/must) $s = 1$

($s \neq 2$, since no $m = 2$ state in old basis)

... gives 3 states: $|s = 1, m = \pm 1, 0\rangle$

$\Rightarrow$ "remaining" 1 base ket of (B)

"must" be $s = 0$

specific example of general "rule":

$j = |j_1 + j_2|$ "to" $|j_1 - j_2|$ in "steps" of 1

$\hookrightarrow$ $|\vec{J}|^2 = |\vec{J}_1|^2 + |\vec{J}_2|^2$ eigenvalue

Here, $j_{1,2} = s_{1,2} = 1/2 \Rightarrow j = |1/2 + 1/2|$ to $|1/2 - 1/2|$

[II]. Find $|s(=0,1), m\rangle$ in terms of $|m_1 m_2\rangle$

(change of basis / unitary transformation)

(1). $\underbrace{|s=1, m=+1\rangle}_{\text{basis }(B)} = \underbrace{|++\rangle}_{\text{old basis }(A)}$

using $\underbrace{m}_{S_z \text{ eigenvalue}} = \underbrace{m_1 + m_2}_{S_{1,2\,z} \text{ eigenvalues}}$;

[See Griffiths sec. 4.4.3 for explicit proof of $|++\rangle$ having $s=1$ eigenvalue, cf. above was "by elimination" ...]

(2). Use lowering operator:

S_- on LHS $= (S_{1-} + S_{2-})$ on RHS,

with $J_\pm |j, m\rangle = \sqrt{(j \mp m)(j \pm m + 1)}\, \hbar\, |j, m \pm 1\rangle$

$$|s=1, m=0\rangle = \frac{1}{\sqrt{2}}\left(|+-\rangle + |-+\rangle\right),$$

not eigenket of $S_{1,2\,z}$

$$\left(\left[|\vec{S}|^2, S_{1,2\,z}\right] \neq 0\right)$$

(3). More lowering (or elimination):

$$|s=1, m=-1\rangle = |--\rangle$$

(4). $|s=0, m=0\rangle = \frac{1}{\sqrt{2}}\left(|+-\rangle - |-+\rangle\right)$

by elimination/orthogonality to above 3 RHS's : HW 9.1 for $s=0$ (under rotation, it is unchanged)

— In general, change of basis:

$$\underbrace{|b^{(\ell)}\rangle}_{\text{new: } |s,m\rangle \text{ here}} = U \underbrace{|a^{(\ell)}\rangle}_{\text{old: } |m_1 m_2\rangle}$$

with matrix element of U in $|a'\rangle$ basis:

$$\langle a^{(k)} | U | a^{(l)} \rangle = \langle a^{(k)} | b^{(l)} \rangle$$

$\Rightarrow$ coefficients on RHS of above 4 relations are unitary operator matrix elements: Clebsch-Gordon (CG) coefficients

— Equivalently: $|\bar{S}|^2$ matrix in (old) $\{m_1, m_2\}$ basis is not diagonal, since $|\bar{S}|^2 = |\bar{S}_1|^2 + |S_2|^2 + 2\,\bar{S}_1 \cdot \bar{S}_2$

$$= \underbrace{|\bar{S}_1|^2 + |\bar{S}_2|^2 + \overbrace{2\, S_{1z} S_{2z}}}_{\substack{\text{diagonal} \\ \text{(in old basis)}}} + \underbrace{S_{1+} S_{2-} + S_{1-} S_{2+}}_{\text{not diagonal}}$$

S_{1+} takes $|-+\rangle$ to $|++\rangle$

$\Rightarrow$ unitary matrix diagonalizing above $|\bar{S}|^2$ in $\{m_1, m_2\}$ basis changes

basis to $\{s, m\}$ ($|\bar{S}|^2$ diagonal):
its elements are $\boxed{\text{CG}}$ coefficients

... generalize to addition of 2
arbitrary angular momenta:

$$\bar{J} \equiv \bar{J}_1 \otimes \mathbb{1} + \mathbb{1} \otimes \bar{J}_2 = \bar{J}_1 + \bar{J}_2;$$

$$[J_i, J_j] = i\varepsilon_{ijk} J_k \;\left(\Rightarrow [|\bar{J}|^2, J_z] = 0\right);$$

$$0 \neq [|\bar{J}|^2, J_{1z}] = -[|\bar{J}|^2, J_{2z}] \quad \left(\text{since } [|\bar{J}|^2, \underbrace{(J_{1z}+J_{2z})}_{J_z}] = 0\right)$$

$$[|\bar{J}|^2, |\bar{J}_1|^2] = 0 = [|\bar{J}|^2, |\bar{J}_2|^2]$$

(generator of infinitesimal rotations for entire system)

$$\mathcal{D}(R) = \mathcal{D}_1(R) \otimes \mathcal{D}_2(R) = \exp\left(\frac{-i\bar{J}_1\cdot\hat{n}\phi}{\hbar}\right) \otimes \exp\left(\frac{-i\bar{J}_2\cdot\hat{n}\phi}{\hbar}\right)$$

(rotation operator for entire system)

Tale of 2 $\boxed{\text{bases}}$

(A) eigenkets of $|\bar{J}_1|^2, |\bar{J}_2|^2, J_{1z}, J_{2z}$

(not of $|\bar{J}|^2$: $[|\bar{J}|^2, J_{1z}] \neq 0 \neq [|\bar{J}|^2, J_{2z}]$)

$$|\vec{J}_{\frac{1}{2}}|^2|j_1\, j_2; m_1, m_2\rangle = j_{\frac{1}{2}}(j_{\frac{1}{2}}+1)\hbar^2|j_1\, j_2; m_1\, m_2\rangle$$

$$J_{\frac{1}{2}z}|j_1\, j_2; m_1\, m_2\rangle = m_{\frac{1}{2}}\hbar|j_1\, j_2; m_1\, m_2\rangle$$

(B). eigenkets of $|\vec{J}|^2$, J_z, $|\vec{J}_1|^2$, $|\vec{J}_2|^2$

(since $[|\vec{J}|^2, |\vec{J}_1|^2] = 0$), but not of

$J_{\frac{1}{2}z}$, since $[|\vec{J}|^2, J_{\frac{1}{2}z}] \neq 0$

$$|\vec{J}_{\frac{1}{2}}|^2|j_1\, j_2; j\, m\rangle = j_{\frac{1}{2}}(j_{\frac{1}{2}}+1)\hbar^2|j_1\, j_2; j, m\rangle$$

$$|\vec{J}|^2|j_1\, j_2; j\, m\rangle = j(j+1)\hbar^2|j_1\, j_2; j\, m\rangle$$

$$J_z|j_1\, j_2; j\, m\rangle = m\hbar|j_1\, j_2; j\, m\rangle$$

CG matrix for $\underset{s_1}{\frac{1}{2}} \oplus \underset{s_2}{\frac{1}{2}} = \underbrace{0, 1}_{s}$

from : $|(s=)1, (m=)+1\rangle = |\underset{m_1}{+}\underset{m_2}{+}\rangle$

$|1, 0\rangle = \frac{1}{\sqrt{2}}\left(|+\underset{m_2=-1/2}{-}\rangle + |-+\rangle\right)$ $\quad = +\frac{1}{2}$

$|1, -1\rangle = |--\rangle$

& $|0, 0\rangle = \frac{1}{\sqrt{2}}\left(|+-\rangle - |-+\rangle\right)$ is

$|--\rangle$

$$\begin{array}{l} \text{row} \sim \text{new} \\ \text{column} \sim \text{old} \\ |s=1, m=-1\rangle \end{array} \quad U^\dagger = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

Again, "$\underbrace{|s, m\rangle}_{\text{new basis}(B)} = \boxed{U} \underbrace{|m_1, m_2\rangle}_{\text{old basis}(A)}$"

$\boxed{\text{Motivation}}$ for computing $\boxed{CG}$:

give amplitude for transition between initial & final state, where one/both sides involve combination of angular momenta,

e.g., spin-0 $\longrightarrow$ two spin-$\frac{1}{2}$ particles
(decay: see Phys 624)

$\underbrace{\text{two spin-}\tfrac{1}{2}\text{ particles}}$
what (precise) combination gives spin-0 (to "match" angular momentum of initial state)

("neglect" orbital angular momentum for simplicity)

or EM transition (Phys 623):

$\langle \text{different spin-}\tfrac{1}{2} \,|\, \text{EM field} \,|\, \text{spin-}\tfrac{1}{2} \rangle$

... (quantize EM) spin-$\frac{1}{2}$ $\longrightarrow$ spin-$\frac{1}{2}$ + photon (spin-1)

(again, combine $\underbrace{\text{spin-}\tfrac{1}{2} \text{ \& spin-}1}_{\text{final}}$ to give $\underbrace{\text{spin-}\tfrac{1}{2}}_{\text{initial}}$)