Last time: used SG experiment to motivate abstract, complex VS containing

(I). **Ket**, $\boxed{|\alpha\rangle}$ describes state of system, e.g., $|S_z;\pm\rangle$

(II). **Bra**, $\langle\alpha|$ dual to ket (usage will be clear later)

$\boxed{\text{Lecture } \boxed{3}}$ (Fri., Sept. 4)

Outline: (III). Operators (denoted by $X, Y \ldots$ in general), some of which are observables ($A, B \ldots$)

- properties in general (will not give/prove all)
- special class (Hermitian: observables) eigenkets as base kets ...
- matrix representation of operators, kets & bras ...
- onto measurements: $\boxed{\text{inner}}$ product crucial ...

(III). $\boxed{\text{Operators}}$ (act on kets from left)

$$X\big(|\alpha\rangle\big) = X|\alpha\rangle \text{ is another ket}$$

... in general ~~of~~ $|\alpha\rangle$

... except $\underbrace{A}_{\text{operator}}|\alpha'\rangle = \underbrace{a'}_{\text{eigenvalue}}\underbrace{|\alpha'\rangle}_{\text{eigenket}}$

e.g. $S_z|S_z;\pm\rangle = \pm\frac{\hbar}{2}|S_z;\pm\rangle$

$$(X+Y)|\alpha\rangle = X|\alpha\rangle + Y|\alpha\rangle$$

- **linear**: $X\big(c_\alpha|\alpha\rangle + c_\beta|\beta\rangle\big) = c_\alpha X|\alpha\rangle + c_\beta X|\beta\rangle$

$(\langle\alpha|)X = \langle\alpha|X$

- (Hermitian) adjoint (definition)

$$X|\alpha\rangle \longleftrightarrow \langle\alpha|X^{\dagger}$$

Hermitian: $X^{\dagger} = X$

- multiplication: $(XY)|\alpha\rangle = XY|\alpha\rangle$

$$\equiv X(Y|\alpha\rangle)$$

don't commute in general: $XY \neq YX$

e.g. S_z, x

associative: $X(YZ) = (XY)Z \ldots$

- $(XY)^{\dagger} = Y^{\dagger}X^{\dagger}$

proof: $XY|\alpha\rangle = X(Y|\alpha\rangle)$

$$\longleftrightarrow \underbrace{(\langle\alpha|Y^{\dagger})}_{\text{dual to } Y|\alpha\rangle} X^{\dagger} = \langle\alpha|Y^{\dagger}X^{\dagger}$$

$|\alpha\rangle|\beta\rangle$ "illegal" (unless different VS)

- "outer" product (legal): $|\beta\rangle\langle\alpha|$ is operator

associative axiom: "can move around · around"

$$(|\beta\rangle\langle\alpha|)\cdot\underbrace{|\gamma\rangle}_{\text{ket}} = \overbrace{|\beta\rangle\cdot\underbrace{\langle\alpha|\gamma\rangle}_{\text{inner product}}}^{\text{ket}}$$

number ↓

$$\langle\alpha|\gamma\rangle \cdot |\beta\rangle \neq \underbrace{\langle\alpha|\left(|\gamma\rangle \cdot |\beta\rangle\right)}_{\text{illegal}}$$

- $X^{+} = |\alpha\rangle\langle\beta|$ if $X = |\beta\rangle\langle\alpha|$

- $$\underbrace{\left(\langle\beta|\right)}_{\text{bra}} \cdot \underbrace{\left(X|\alpha\rangle\right)}_{\text{ket}} = \left(\langle\beta|X\right) \cdot \left(|\alpha\rangle\right)$$

$$= \underbrace{\langle\beta|X|\alpha\rangle}_{\text{number}}$$

$$\boxed{\langle\beta|X|\alpha\rangle = \langle\alpha|X^{+}|\beta\rangle^{*}}$$

Onto Hermitian operators $(A, B \ldots)$

Theorem : (1). $\boxed{\text{eigenvalues are real}}$

(2) $\boxed{\text{eigenkets orthogonal}}$

$$\langle a''|A|a'\rangle = \langle a''|a'|a'\rangle \longleftrightarrow \langle a''|\overset{A}{\overset{\parallel}{A^{+}}}|a'\rangle$$

$$= a'\langle a''|a'\rangle \qquad\qquad = a''^{*}\langle a''|a'\rangle$$

$$\boxed{0 = (a' - a''^{*})\langle a''|a'\rangle}$$

(1) $a' = a''$ (use $\langle a'|a'\rangle > 0$)

$\Rightarrow a' = a'^{*}$ (a real)

(2) $a' \neq a'' = a''^{*} \Rightarrow (a' - a''^{*}) \neq 0$

$\Rightarrow \langle a'' | a' \rangle = 0$

e.g. $S_z |S_z ; \pm\rangle = \pm \frac{\hbar}{2} |S_z ; \pm\rangle$

$|S_z ; \pm\rangle \equiv |\pm\rangle \Rightarrow \boxed{\langle + | - \rangle = 0}$

$$\boxed{\langle a'' | a' \rangle = \delta_{a'' a'}}$$

choose $\langle \pm | \pm \rangle = 1$

... Hermitian operators are observables

Postulate : (N) eigenkets of A form a complete set (base kets) for N dimensional VS

$\langle a' | \alpha \rangle = \langle a' | \sum_{a''} c_{a''} | a'' \rangle \Rightarrow \boxed{c_{a'} = \langle a' | \alpha \rangle}$

$$\boxed{|\alpha\rangle = \sum_{a'} |a'\rangle \langle a' | \alpha \rangle}$$

closure/completeness: $\boxed{\mathbb{1} = \sum_{a'} |a'\rangle\langle a'|}$

$$\langle \alpha | \alpha \rangle = \langle \alpha | \sum_{a'} |a'\rangle\langle a'| \alpha \rangle$$

$$\boxed{= \sum_{a'} |\langle a' | \alpha \rangle|^2 = 1} \quad \text{(normalized)}$$

e.g. $\boxed{\mathbb{1} = |+\rangle\langle +| + |-\rangle\langle -|}$ & $|S_x ; \pm\rangle = \frac{1}{\sqrt{2}} (|+\rangle \pm |-\rangle)$

Note : $S_z|S_x;+\rangle = |S_x;-\rangle \cancel{\propto} |S_x;+\rangle$

2nd lecture below

Projection operator : $\Lambda_{a'} = |a'\rangle\langle a'|$

$$\Lambda_{a'}|\alpha\rangle = |a'\rangle\underbrace{\langle a'|\alpha\rangle}_{c_{a'}} = \boxed{c_{a'}|a'\rangle}$$

$$\boxed{\sum_{a'}\Lambda_{a'} = \mathbb{1}}$$

$$\boxed{A} = \sum_{a'}\sum_{a''}|a''\rangle\underbrace{\langle a''|A|a'\rangle}_{= \delta_{a'a''}a'}\langle a'| = \sum_{a'} a'|a'\rangle\langle a'| = \boxed{\sum_{a'} a'\Lambda_{a'}}$$

e.g. $\boxed{S_z = \frac{\hbar}{2}\left(|+\rangle\langle +| - |-\rangle\langle -|\right)}$

Matrices (use eigenkets of A as basekets

row, column

$$X = \sum_{a'}\sum_{a''}|a''\rangle\langle a''|X|a'\rangle\langle a'|$$

$\to N^2$ numbers

$$X \doteq \begin{pmatrix} \langle a^{(1)}|X|a^{(1)}\rangle & \langle a^{(1)}|X|a^{(2)}\rangle & \cdots \\ \langle a^{(2)}|X|a^{(1)}\rangle & \cdots & \\ \vdots & & \end{pmatrix}$$

X matrix different if basekets of B used

$\boxed{\text{check}}$ $\langle a''|X|a'\rangle = \langle a'|X^{\dagger}|a''\rangle^{*}$

$X^{\dagger}$ represent by complex conjugated transpose of (matrix) X

Hermitian : $\langle a''|B|a'\rangle = \langle a'|B|a''\rangle^{*}$

– $\boxed{Z = XY}$ ...

– $\boxed{\text{ket}}$ $\langle a'|\gamma\rangle = \langle a'|X|\alpha\rangle$

$$= \sum_{a''} \underbrace{\langle a'|X|a''\rangle}_{\text{"matrix" } X} \langle a''|\alpha\rangle$$

$$|\alpha\rangle \doteq \begin{pmatrix} \langle a^{(1)}|\alpha\rangle \\ \vdots \\ \langle a^{(N)}|\alpha\rangle \end{pmatrix}$$

ket column

... $\langle\gamma| \doteq (\langle a^{(1)}|\gamma\rangle^{*} \; \ldots \; \langle a^{(N)}|\gamma\rangle^{*})$

(bra) row

$$\text{inner product} = \langle\beta|\alpha\rangle = \sum_{a'} \underbrace{\underbrace{\langle\beta|a'\rangle}_{\langle a'|\beta\rangle^{*}}}_{\text{row}} \underbrace{\langle a'|\alpha\rangle}_{\text{column}}$$

... $\boxed{|\beta\rangle\langle\alpha|}$ square matrix ...

$\boxed{\text{e.g.}}$ spin-$1/2$ system : 1^{st} row/column index highest spin component

$|+\rangle \doteq \begin{pmatrix} 1 \\ 0 \end{pmatrix}$; $|-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \langle +|-\rangle \\ \langle -|-\rangle \end{pmatrix}$

$\left[= \begin{pmatrix} \langle +|+\rangle \\ \langle -|+\rangle \end{pmatrix} \right]$ $\boxed{S_z} \doteq \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\boxed{= \hbar/2 \; \sigma_3}$

$\boxed{\text{Measurements}}$: $\boxed{|\alpha\rangle = \sum_{a'} c_{a'} |a'\rangle}$

postulate $\boxed{1}$ measurement of A (observable) "throws" system into one of eigenkets of A

$$|\alpha\rangle \xrightarrow{A \text{ measurement}} |a'\rangle$$

$\boxed{A \text{ is measured to be } a'}$

e.g. Ag atom after SG$\hat{z}$ is in either $|\pm\rangle$

postulate $\boxed{2}$ probability for a'

$$= |\langle a'|\alpha\rangle|^2 = |c_{a'}|^2$$

determine probabilities by measuring A on $\boxed{\text{ensemble}}$ of identical systems, e.g. Ag beam coming out of SG$\hat{z}$ with $|-\rangle$

beam $\boxed{\text{blocked}}$ is all $\left| S_z ; + \right\rangle$ (or $|+\rangle$)

... selection of $|+\rangle$ represented by

$\boxed{\text{projection operator}}$: $\Lambda_{a'} |\alpha\rangle$

$$= |a'\rangle\langle a'|\alpha\rangle$$

resulting ket

... $\boxed{\text{reasonable}}$ postulates :

For system is in $|a'\rangle$,

probability to give a'' is 0 ... follows

from $\boxed{\langle a' | a'' \rangle = 0}$ ↑ expectation

& probabilities add up to 1 (normalized)

$$\boxed{\sum_{a'} |\langle a' | \alpha \rangle|^2 = 1}$$

Expectation value of $\boxed{A}$: average

of measurement done on ensemble

$$\boxed{\langle A \rangle_\alpha} \overset{\text{definition}}{\equiv} \langle \alpha | A | \alpha \rangle$$

$$= \sum_{a''} \sum_{a'} \langle \alpha | a'' \rangle \underbrace{\langle a'' | A | a' \rangle}_{a' \delta_{a'a''}} \langle a' | \alpha \rangle$$

$$= \sum_{a'} a' |\langle a' | \alpha \rangle|^2$$

a' ← measured value; $|\langle a' | \alpha \rangle|^2$ ← probability

$\Rightarrow \langle A \rangle_\alpha$ is $\boxed{\text{indeed}}$ "average" ...

Note: $\underbrace{\text{eigenvalues}}_{\pm \hbar/2 \text{ only for } S_z}$ vs. $\underbrace{\text{expectation values}}_{\boxed{\text{any}} \text{ (real) value } \boxed{\text{between}} \pm \hbar/2}$

Next, use above ideas to "prove" superpositions of $|S_z; \pm\rangle$ in $|S_{x,y}; \pm\rangle$ that we had guessed earlier