Lecture $\boxed{25}$, Oct. 28 (Wed.)

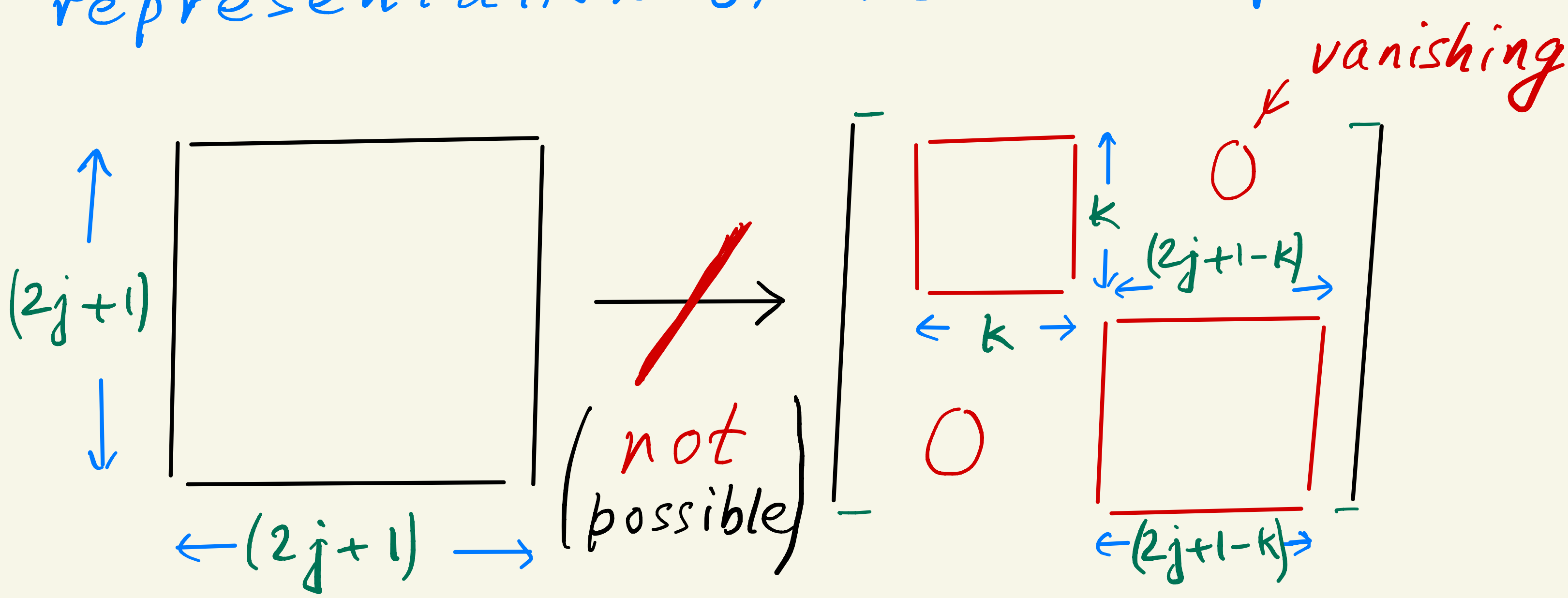
Outline for today (& Fri.)

- Rotation matrices for general j
  - $j = 1$ using Euler angles
- start orbital angular momentum & spherical harmonics: from "rotations" viewpoint

General points

$$\mathcal{D}^{(j)}_{m'm}(R) \text{ form } (2j+1) \times \overbrace{(2j+1)}^{\text{number of } m \text{ values}} \text{ matrix:}$$

$(2j+1)$-dimensional irreducible representation of rotation operator



$\Rightarrow$ General rotation matrix ("mixture" of j-values):

vanishing

$$\begin{bmatrix} \boxed{j_1} & 0 & 0 \\ 0 & \boxed{j_2} & 0 \\ 0 & 0 & \boxed{j_3} \end{bmatrix}$$

$\boxed{}$ is $(2j+1)\times(2j+1)$ matrix $\mathcal{D}^{(j)}_{m'm}$

— rotation matrices : $\mathcal{D}^{(j)}_{m',m}(R)$ for given j form group

varied

– Recall group theory for rotations:

– R's (3×3) orthogonal matrices (rotations for 3d-vectors in real space) form $SO(3)$ group (classical)

– Onto QM : $\mathcal{D}(R)$ (rotation operator acting on kets in abstract space) "inherit" properties of R ...

- Concretely, spin-$\frac{1}{2}$: $\mathcal{D}(R)$ matrices are $\Sigma \equiv \exp\left(-\frac{i\vec{\sigma}\cdot\hat{n}\phi}{2}\right)$ form $SU(2)$ group (matrices acquire properties of operators)

- generalize to $j > \frac{1}{2}$:
- identity $(2j+1)\times(2j+1)$ matrix : "rotation" with $\phi = 0$
- inverse is $-\phi$ (same $\hat{n}$)
- product: (of matrices) $\sum_{m'} \mathcal{D}^{(j)}_{m''m'}(R_1)\,\mathcal{D}^{(j)}_{m'm}(R_2) = \mathcal{D}^{j}_{m''m}(R_1 R_2)$
- $\mathcal{D}^{(j)}_{m'm}(R)$ matrix is unitary (like operator) : $\mathcal{D}^{(j)}_{m'm}(R^{-1}) = \mathcal{D}^{*(j)}_{m\,m'}(R)$
- What does $\mathcal{D}^{(j)}_{m'm}(R)$ "give us"?

Rotated state, $\underbrace{\mathcal{D}(R)}_{\text{operator}}|j,m\rangle$

$$= \sum_{m'} \underbrace{|j,m'\rangle\langle j,m'|}_{\mathbb{1}}\,\mathcal{D}(R)\,|j,m\rangle$$

matrix element

$|$ fixed $j, m\rangle$ form complete set since $\mathcal{D}(R)$ doesn't connect $j \neq j'$

$\Rightarrow$ matrix element $\boxed{\mathcal{D}^{(j)}_{m'm}(R)}$ is amplitude to find rotated version of $|j, m\rangle$ being $|j, m'\rangle$

– Onto more $\boxed{\text{concrete}}$ $\mathcal{D}^{(j)}_{m'm}(R)$:

use Euler angles : matrix version of

$$\mathcal{D}(\alpha, \beta, \gamma) = \mathcal{D}_z(\alpha)\, \mathcal{D}_y(\beta)\, \mathcal{D}_z(\gamma):$$

$\boxed{\text{(follows from } R \text{ relation)}}$

$\mathcal{D}^{(j)}_{m'm}(\alpha, \beta, \gamma)$ (done already for $j = 1/2$)

$$\langle j, m' | \exp\left(\frac{-iJ_z\alpha}{\hbar}\right) \exp\left(\frac{-iJ_y\beta}{\hbar}\right) \exp\left(\frac{-iJ_z\gamma}{\hbar}\right) | j, m\rangle$$

$\exp(-im'\alpha)$ (check) $\qquad \exp(-im\gamma)$

$$= \exp[-i(m'\alpha + m\gamma)] \langle j, m' | \exp(-iJ_y\beta/\hbar) | j, m\rangle$$

Rotations about z-axis (on 2 sides) "diagonal"; only middle rotation (about y-axis) connects $m' \neq m$

- So, work on it:

$$\boxed{d^{(j)}_{m'm}(\beta) = \langle j, m' | \exp -\frac{i J_y \beta}{\hbar} | j, m \rangle}$$

- Reminder: $d^{(j=1/2)} = \begin{pmatrix} \cos\beta/2 & -\sin\beta/2 \\ \sin\beta/2 & \cos\beta/2 \end{pmatrix}$

step back

$$= \exp\left(-\frac{i\sigma_2\beta}{2}\right) = \sum_{n=0}^{\infty}\left(-i\sigma_2\beta/2\right)^n$$

$$= \cos\beta/2 \; \mathbb{1} - i\sigma_2 \sin\beta/2 \quad \underbrace{\left(\text{only 2 "terms/structures"}\right)}$$

since $\sigma_2^{\text{even}} = \mathbb{1}$ and $\sigma_2^{\text{odd}} = \sigma_2$

- Onto $j=1$: matrix representation of $J_y = (J_+ - J_-)/(2i)$ (matrix elements of $J_\pm$ earlier)

3-dimensional ("analog" of σ_2 for $j = 1/2$: S_y)

- Use $\langle j', m' | J_\pm | j, m \rangle$ of before :

$$J_y^{(j=1)} = \frac{\hbar}{2} \begin{array}{cc} \begin{array}{ccc} m=1 & m=0 & m=-1 \end{array} & \\ \begin{pmatrix} 0 & -\sqrt{2}i & 0 \\ \sqrt{2}i & 0 & -\sqrt{2}i \\ 0 & \sqrt{2}i & 0 \end{pmatrix} & \begin{array}{c} m'=1 \\ m'=0 \\ m'=-1 \end{array} \end{array}$$

[off-diagonal, since $|j, m\rangle$ are not J_y eigenkets]

Use $d^{(j=1)} = \sum_{n=0}^{\infty} \left[-i J_y^{(j=1)} \beta / \hbar \right]^n$

$$= \mathbb{1} - i \frac{J_y^{(j=1)} \beta}{\hbar} + \frac{1}{2!} \left[\frac{-i J_y^{(j=1)} \beta}{\hbar} \right]^2 + \ldots$$

- Can $\left[J_y^{(j=1)} \right]^2$ be "simplified" in terms of $\mathbb{1}$ and $\left[J_y^{(j=1)} \right]^1$ ("like" $\sigma_2^2 = \mathbb{1}$)? $\boxed{\text{NO}}$

— choose diagonal basis : $J_y^{(j=1)} \propto \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix}$

(J_y has same eigenvalues as J_z, but eigenkets different)

$$\Rightarrow \left[J_y^{(j=1)}\right]^2 \propto \begin{pmatrix} 1 & & \\ & 0 & \\ & & +1 \end{pmatrix}$$

$$\not\propto \underbrace{\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}}_{\mathbb{1}} \text{ "}+\text{" } \underbrace{\begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix}}_{\left[J_y^{(j=1)}\right]^1} \ ?$$

(not possible)

$-$ $\left[J_y^{(j=1)}\right]^2$ independent structure...

... but $\left[J_y^{(j=1)}\right]^3 \propto \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix}$

$= \left[J_y^{(j=1)}\right]^1$ ("like" $\sigma_2^3 = \sigma_2$)

... and so on: $\left[J_y^{(j=1)}\right]^4 = \underbrace{\left[J_y^{(j=1)}\right]^3}_{\left[J_y^{(j=1)}\right]^1}\left[J_y^{(j=1)}\right]^1$

(See HW 7.2 for details)

$$\exp\left(-iJ_y^{(j=1)}\beta/\hbar\right) = \mathbb{1} - \left(\frac{J_y^{(j=1)}}{\hbar}\right)^2 (1-\cos\beta)$$

$$- i\left(J_y^{(j=1)}/\hbar\right)^1 \sin\beta$$

Orbital angular momentum, $\bar{L}$

— can start "without rotations":

$\bar{L} = \bar{x} \times \bar{p}$ (operators) ... $\boxed{\text{vs.}}$

here, $\bar{J}$ (angular momentum in general: orbital or spin) "defined" as generator of infintesimal rotations

— goal: connect above two viewpoints by showing

(i) $\bar{L}$ $(= \bar{x} \times \bar{p})$ satisfies $[J_i, J_j] = i\varepsilon_{ijk} J_k$

(ii) $\bar{L}$ $(= \bar{x} \times \bar{p})$ rotates position eigenket infinitesimally (generates rotations)

["like" treatment of spin:
to begin with, description of
2-state system (empirically, e.g.,
Stern-Gerlach experiment)... then
$[S_i, S_j] = i\varepsilon_{ijk} S_k$ ($\bar{J}$ commutation
relations) & $\bar{S}$ "rotates" kets...]

— Then ("about turn"): "forget"
$\bar{L} = \bar{x} \times \bar{p}$, using $\bar{L}$ as generator
of rotations ("like" $\bar{J}$) to figure
out $\bar{L}$'s action on wavefunction
(position-space representation of
ket) ... recovering $|\bar{L}|^2 \sim$ angular
part of ∇^2... knew already
from $\bar{L} = \bar{x} \times \bar{p}$