Lecture $\boxed{24}$, Oct. 26 (Mon.)

Outline for today (& Wed.)

- continue obtaining eigenvalues & eigenstates of angular momentum
- matrix elements of angular momentum operators (generators of rotations)
- Matrix representations of rotation operators

(done already for spin-$\frac{1}{2}$: lowest dimensional realization of angular momentum)

(More systematic/mathematical treatment ...)

Why are $J_\pm = J_x \pm i J_y$ raising/lowering operators?

$$J_z \left(J_\pm |a,b\rangle\right) = \left(\underbrace{[J_z, J_\pm]}_{[\text{use } (4)]\ \pm\hbar J_\pm} + J_\pm \overset{b}{J_z}\right)|a,b\rangle$$

$$= (b \pm \hbar)\left(J_\pm |a,b\rangle\right) \quad \ldots(5)$$

(J_z eigenvalue increased/reduced by $\hbar$)

[a bit like SHO: $a^\dagger$, a increase/reduce energy by 1 unit]

– Eigenvalue of $|\vec{J}|^2$ unchanged:

$$|\vec{J}|^2 \left(J_\pm |a,b\rangle\right) = J_\pm |\vec{J}|^2 |a,b\rangle = a\, J_\pm |a,b\rangle$$

using $[J_\pm, |\vec{J}|^2] = 0$ [Eq. (4)] $\quad \ldots(6)$

– $J_\pm |a,b\rangle$ simultaneous eigenkets of J_z & $|\vec{J}|^2$: $J_\pm |a,b\rangle = \underset{\text{normalization}}{c_\pm} |a, \underbrace{b\pm\hbar}_{\text{raise/lower } J_z}\rangle$

($|\vec{J}|^2$ unchanged)

Relating a (eigenvalue of $|\vec{J}|^2$) & b (eigenvalue of J_z): (I)

— applying J_+ repeatedly to increase J_z eigenvalue (for fixed a) indefinitely? $\boxed{\text{No}}$

Intuition: J_z^2 "$\subset$" $|\vec{J}|^2 = \sum J_i^2$, $i=1,2,3$,

so "expect" $\boxed{b^2 \leq a}$

Proof: Use $\left(|\vec{J}|^2 - J_z^2\right) = J_x^2 + J_y^2 = \frac{1}{2}\left(J_+ J_- + J_- J_+\right)$
$= \frac{1}{2}\left(J_+^\dagger J_+ + J_+ J_+^\dagger\right)$

and $\left(\langle \alpha | X^\dagger\right)\left(X | \alpha\rangle\right) = \langle \beta | \beta \rangle \geq 0 \left(|\beta\rangle \equiv X|\alpha\rangle\right)$

choose J_+ or $J_+^\dagger$

$\Rightarrow \langle a, b | \left(|\vec{J}|^2 - J_z^2\right) | a, b\rangle \geq 0$ ($J_z^2 \to b^2$)

$\Rightarrow a \geq b^2 \Rightarrow \boxed{J_+ |a, b_{max}\rangle = 0}$ $(b < b_{max})$

(a bit like lowering ... for SHO)

Relating a to b : (II)

$$J_-\underbrace{\left(J_+|a,b_{max}\rangle\right)}_{0} = 0$$

Use $J_-J_+ = |\vec{J}|^2 - J_z^2 - \hbar J_z$:

$$\big(\underset{a}{|\vec{J}|^2} - \underset{b^2_{max}}{J_z^2} - \hbar \underset{b_{max}}{J_z}\big)\underbrace{|a,b_{max}\rangle}_{\neq 0} = 0$$

$$\Rightarrow a - b^2_{max} - \hbar b_{max} = 0$$

$$b^2_{max} + \hbar b_{max} - a = 0$$

solved by

$$b_{max} = \frac{1}{2}\left(-\hbar \pm \sqrt{\hbar^2 + 4a}\right) \quad \dots (7)$$

— Similarly, b_{min} : $J_-|a,b_{min}\rangle = 0$ $(b > b_{min})$

$\Rightarrow$ ca<u>n't</u> lower/raise more than b_{min} / b_{max}

- Use $J_+ \left(J_- |a, b_{min}\rangle \right) = 0$

and $J_+ J_- = |\vec{J}|^2 - J_z^2 + \hbar J_z$:

$$a - b_{min}^2 + \hbar b_{min} = 0$$

$$b_{min}^2 - \hbar b_{min} - a = 0$$

solved by

$$b_{min} = \frac{1}{2}\left(+\hbar \pm \sqrt{\hbar^2 + 4a} \right) \quad \cdots (8)$$

- Comparing Eqs. (7), (8) and requiring $b_{max} > b_{min}$:

[pick "+" in Eq. (7) and "−" in Eq. (8)]: $\boxed{b_{max} = -b_{min}}$

$$= \left(-\hbar + \sqrt{\hbar^2 + 4a} \right) / 2$$

$$\Rightarrow \underbrace{-b_{max}}_{b_{min}} \leq b \leq b_{max} \quad \ldots (9)$$

Now,

$$\text{--}j\quad |a, b_{max}\rangle \propto (J_+)^{n} |a, b_{min}\rangle$$

$n \geq 0$, integer

(can "reach" b_{max} from b_{min} using successive J_+)

$$\Rightarrow b_{max} = b_{min} + n\hbar \ldots (10) \Rightarrow 2b_{max} = n\hbar$$

$$\boxed{b_{max} = n\hbar/2} : \boxed{j} \equiv \frac{b_{max}}{\hbar} = \boxed{n/2}$$

maximum value of J_z eigenvalue

j integer or half (n = even/odd)

– $\boxed{m \equiv b/\hbar}$ $\Rightarrow m = \underbrace{-j, -j+1 \ldots j-1, +j}$

general eigenvalue of J_z in $\hbar$ units

go thru' $m = 0$ for integer j (not for half...)

– Using (7) (with "+") and $b_{max} = j\hbar$ gives

$$\boxed{a = j(j+1)\hbar^2} \text{ (eigenvalue of } |\vec{J}|^2)$$

– Use $|j, m\rangle$ for $|a, b\rangle$:

$$|\vec{J}|^2 |j, m\rangle = j(j+1)\hbar^2 |j, m\rangle \quad j = \text{integer or half}$$

$$J_z |j, m\rangle = m\hbar |j, m\rangle \quad m = -j, -j+1 \ldots j-1, j$$

Matrix elements of angular momentum operators

- So far, $\underbrace{\langle j', m'|}_{\text{normalized}} |\vec{J}|^2 |j, m\rangle = \delta_{jj'}\,\delta_{mm'}\, j(j+1)\hbar^2$

and $\langle j', m' | J_z | j, m\rangle = \delta_{jj'}\,\delta_{mm'}\, m\hbar$

$(|j', m'\rangle, |j, m\rangle)$ orthogonal due to $|\vec{J}|^2, J_z$ Hermitian

- Onto $J_\pm$: $\langle j, m | J_+^\dagger \underbrace{J_+ | j, m\rangle}_{c^+_{jm} |j, m+1\rangle}$

$$= |c^+_{jm}|^2 = \langle j, m | \underbrace{|\vec{J}|^2 - J_z^2 - \hbar J_z}_{\boxed{J_- J_+ : \text{see before Eq.(7)}}} | j, m\rangle$$

$$= \hbar^2 [j(j+1) - m^2 - m]$$

$$\Rightarrow J_+ | j, m\rangle = \hbar\sqrt{(j-m)(j+m+1)}\, |j, m+1\rangle$$

- Similarly, $J_- | j, m\rangle = \hbar\sqrt{(j+m)(j-m+1)}\, |j, m-1\rangle$

$$\Rightarrow \langle j', m' | J_\pm | j, m\rangle = \sqrt{(j \mp m)(j \pm m + 1)}\,\hbar\, \delta_{jj'}\, \delta_{m', m\pm 1}$$

Onto Matrix elements of rotation operators (using those of their generators, $\bar{J}$ gotten above)

- Wigner functions (in general: spin-$\frac{1}{2}$ earlier)

$$\mathcal{D}^{(j)}_{m'm}(R) \equiv \langle j, m' | \underbrace{\exp\left(-\frac{i\bar{J}\cdot\hat{n}\phi}{\hbar}\right)}_{\mathcal{D}(R)} | j, m\rangle$$

(matrix element) $\hookrightarrow$ parametrized by $\boxed{(\hat{n}, \phi)}$

- again, for spin-$\frac{1}{2}$, these are $\boxed{\Sigma} = \exp\left(-\frac{i\bar{\sigma}\cdot\hat{n}\phi}{2}\right)$

- Can $\mathcal{D}(R)$ "connect" ket & bra with different j values? $\boxed{\text{No}}$

$$\left[|\bar{J}|^2, \text{ any function of } J_k\right] = 0$$

e.g., $\mathcal{D}(R)$

$$\Rightarrow |\bar{J}|^2 \left(\mathcal{D}(R)|j, m\rangle\right) = \mathcal{D}(R)\left(|\bar{J}|^2|j, m\rangle\right)$$

$$= j(j+1)\hbar^2 \left(\mathcal{D}(R)|j, m\rangle\right)$$

$\mathcal{D}(R)|j, m\rangle$ still eigenket of $|\bar{J}|^2$ with same eigenvalue (j, $j'(\neq j)$ orthogonal)

[Rotation cannot change j value, but will change m (in general)]