Lecture 23, Oct. 23 (Fri.)

Outline for today / next week

- Rotation matrices (Σ's) for spin-$\frac{1}{2}$ form SU(2) group
- relate SU(2) to SO(3) of R's
- Euler angles parametrization of rotations (from classical mechanics: rigid body): apply to spin-$\frac{1}{2}$ in QM
- Onto eigenvalues/eigenstates of angular momentum (higher than dimension 2 of spin-$\frac{1}{2}$ done so far):

$$|\vec{J}|^2 |j, m\rangle = j(j+1)\hbar^2 |j, m\rangle$$

$$J_z |j, m\rangle = m\hbar |j, m\rangle$$

where j = integer or half-integer (>0)

& $m = -j, -j+1 \ldots j-1, j$ ...

Rotation matrices for spin-$\frac{1}{2}$

$\Sigma = \exp\left(-\frac{i\bar{\sigma}\cdot\hat{n}\phi}{2}\right)$ is 2×2 comblex unitary matrix

defined as power series

[check : $\Sigma^{+} = \exp\left(+\frac{i\bar{\sigma}^{+}\cdot\hat{n}\phi}{2}\right)$

$= \exp\left(+\frac{i\bar{\sigma}\cdot\hat{n}\phi}{2}\right)$, using $\sigma_i^{+} = \sigma_i$

Use $\exp(iG\lambda)\, A \exp(-iG\lambda) = A +$ (Baker-Hausdorff) $i\lambda[G,A] + \left(\frac{i^2\lambda^2}{2!}\right)[G,[G,A]] + \ldots$

with $A = \mathbb{1}$ and $G\lambda = \bar{\sigma}\cdot\hat{n}\phi/2 \ldots$

to give $\Sigma^{+}\Sigma = \mathbb{1}$]

— $\det(\Sigma) = 1 \Rightarrow S\,U(2)$

special unitary dimension

[check : $\det\Sigma = \det\left[\exp(-i\bar{\sigma}\cdot\hat{n}\phi/2)\right] =$
$\exp\left[-i\,\boxed{\text{trace}}\text{ of }(\bar{\sigma}\cdot\hat{n}\phi/2)\right] = 1$ (σ's traceless)

— "Independent" of rotations, general

SU(2) matrix: $U(a,b) = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$, with

$|a|^2 + |b|^2 = 1$

[check both ways: above U is unitary, with determinant $+1$ ... and any SU(2) matrix can be recast in above form]

- Connect... Σ can be written as $U(a,b)$:

$\mathrm{Re}\, a = \cos\phi/2$; $\mathrm{Im}\, a = -n_z \sin\phi/2$;

$\mathrm{Re}\, b = -n_y \sin\phi/2$ & $\mathrm{Im}\, b = -n_x \sin\phi/2$

- $\boxed{\text{Flipped}}$ most general SU(2) matrix can be rotation: given a, b, "invert" above to get $\boxed{\phi, \hat{n}}$

- (Sanity) check: match number of independent (real) parameters

Σ has 3 $(\hat{n}, \phi)$

U has $4\,(\underbrace{a, b}_{\text{complex}}) - \underbrace{1 \text{ condition}}_{|a|^2+|b|^2=1} = 3$

– Check U's form group:

$$U^{-1} = U^{\dagger} \quad (U^{\dagger}U = \mathbb{1})$$

$$(U_1 U_2)^{\dagger}(U_1 U_2) = U_2^{\dagger} \underbrace{U_1^{\dagger} U_1}_{\mathbb{1}} U_2$$

$$= U_2^{\dagger} U_2 = \mathbb{1}$$

– Is there 1-to-1 correspondence between R's of $SO(3)$ & Σ's of $SU(2)$ [both have 3 real parameters]

$\boxed{2\pi \text{ \& } 4\pi \text{ rotations}}$: $\mathbb{1}$ for R, vs $-\mathbb{1}$ & $+\mathbb{1}$ for Σ

– So, $\boxed{2}$-to-1 correspondence:

[$U(-a, -b)$ & $U(a, b)$ map to same R]

– locally $SO(3)$ isomorphic to $SU(2)$:

if $\Sigma \to R$, then $\Sigma + \delta\Sigma \to \underbrace{R + \delta R}_{\text{unique}}$

$\boxed{\text{Euler angles}}$ ("classical version")

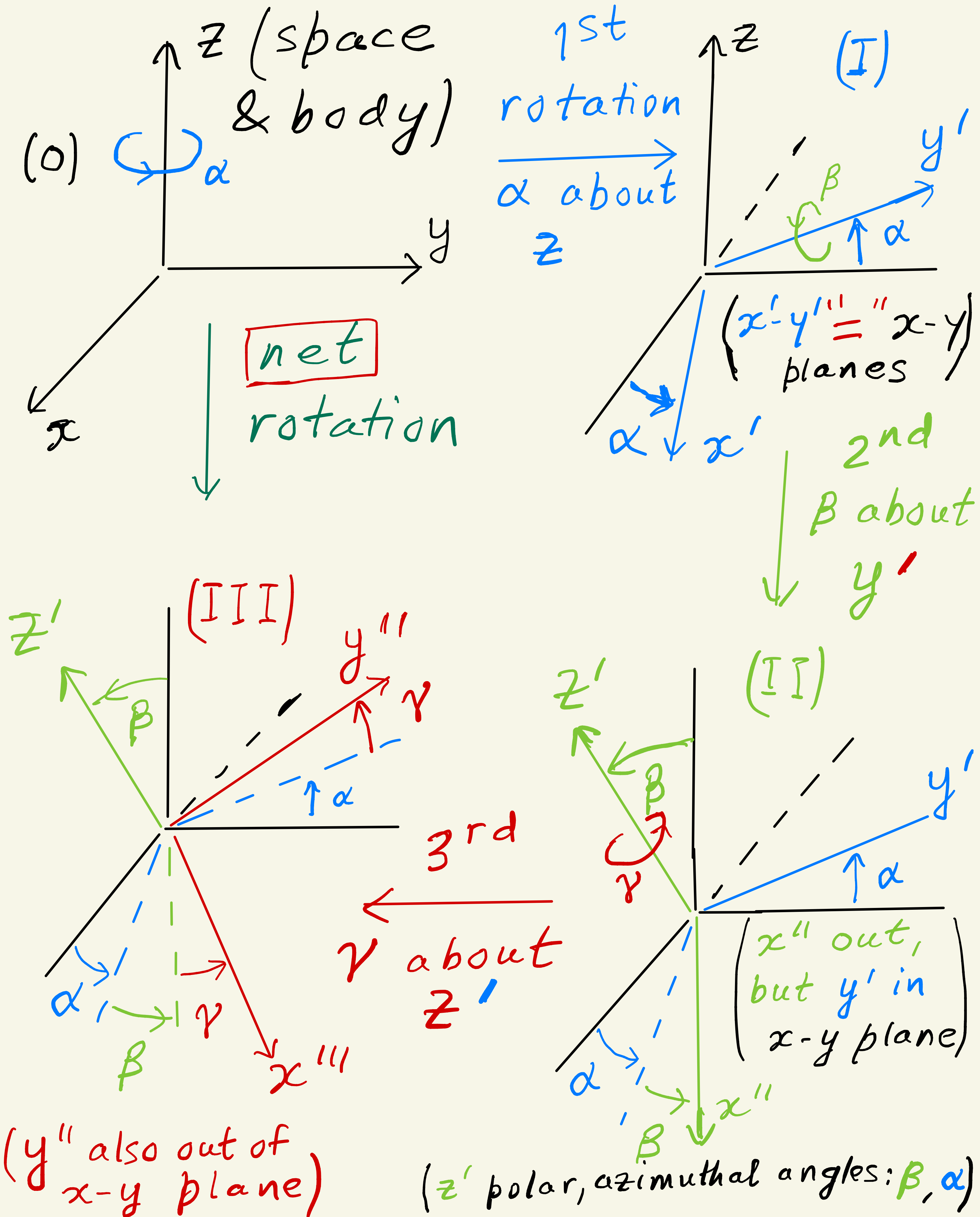
- another way to parametrize rotations [in 3 steps, each being rotation (angles α, β, γ)]
- (Rigid) body (fixed relative to body) axes coincide with space ("absoutely" fixed) axes initially ... but not later (i.e., body axes moving relative to (different than) space axes
- $\boxed{\text{Goal}}$ (Goldstein: sec 4.4, p. 150...) ... (a)

$$\underbrace{R(\alpha, \beta, \gamma)}_{\text{(net) rotation}} = R_{z'}(\gamma)\, R_{y'}(\beta)\, R_z(\alpha)$$

(3 x 3 orthogonal matrix)

axis angle

cf. about x' in Goldstein



(0)
z (space & body)
α
y
x
1st rotation
α about z
(I)
z
y'
β
α
(x'-y' "=" x-y planes)
α
x'
net rotation
2nd
β about y'
(II)
z'
β
y'
γ
α
(x'' out, but y' in x-y plane)
α
β
x''
(z' polar, azimuthal angles: β, α)
(III)
z'
y''
γ
β
α
3rd
γ about z'
α
β
γ
x'''
(y'' also out of x-y plane)

Euler angles: modified (QM) version

"Problem": $R_{y', z'}$ about body axes; but simpler formulae for $\bar{S}$ ($\bar{L}$) about space axes...

so convert: $\boxed{\text{Claim}}$ (buckle up for proof!)

$$\boxed{R(\alpha, \beta, \gamma) = R_z(\alpha) R_y(\beta) R_z(\gamma)}$$

[2nd rotation in (a)]

step 1: $R_{y'}(\beta)$, can be written as

$$\underbrace{R_z(\alpha)}_{\substack{\text{rotate about space } z\text{:} \\ \text{body } z \text{ azimuth} \to \alpha \\ \text{\& body } y \to y'}} \quad \underbrace{R_y(\beta)}_{\substack{\text{rotate about } y\text{:} \\ \text{body } z \\ \text{polar } \beta, \\ \text{azimuth } 0}} \quad \underbrace{R_z^{-1}(\alpha)}_{\substack{\text{"undo" 1st rotation:} \\ y' \to y}} \quad \ldots (b)$$

$\boxed{\text{Proof}}$ $\quad (I) \xrightarrow[\text{LHS of (b)}]{} (II)$

(See figure on next page)

$\big\downarrow$ RHS of (b)

(1) body y axis as in (II): y'

(2) body z-axis has polar β, azimuth α, as in (II): z'

$\Rightarrow$ (3) body x-axis as in (II)

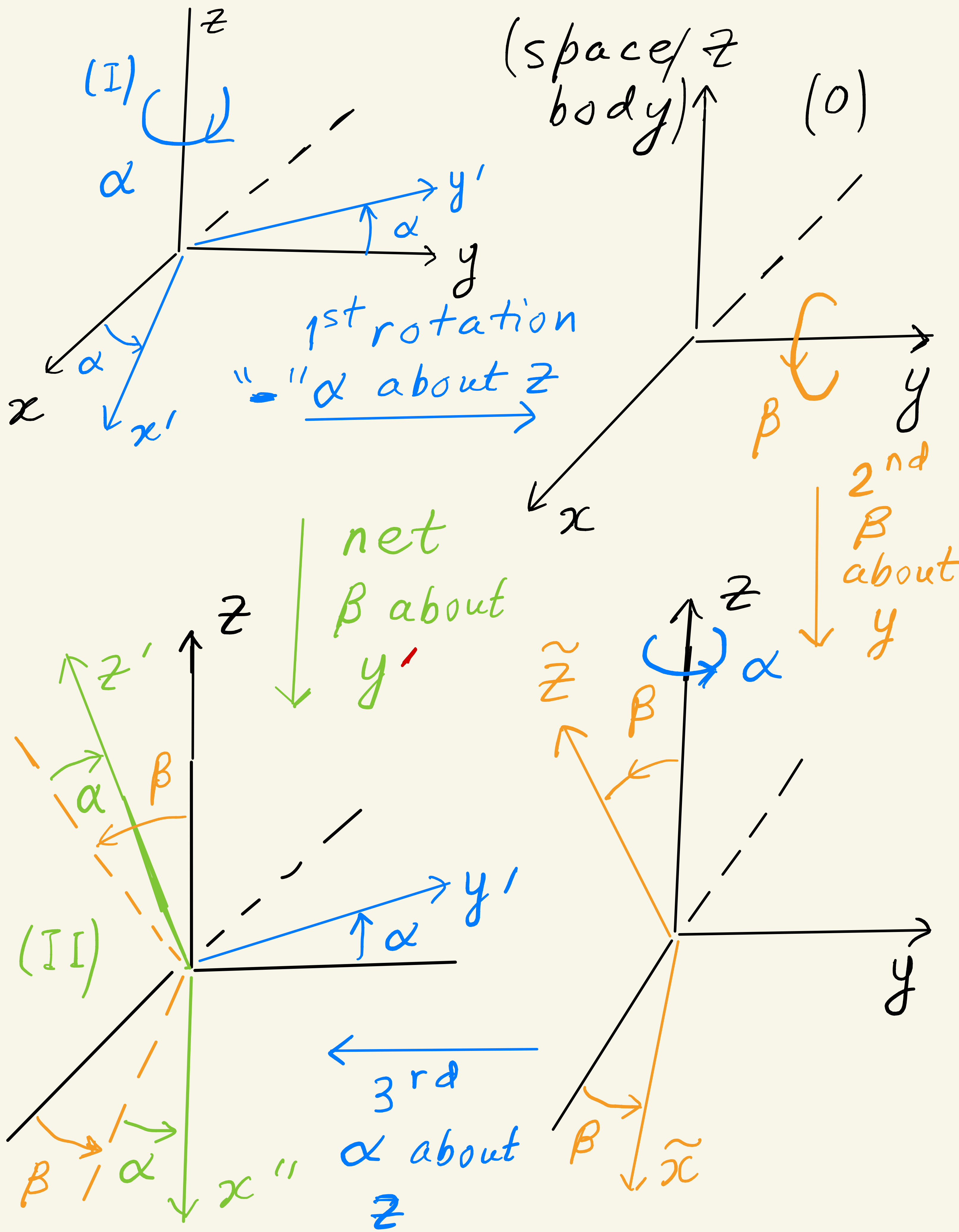
— So, RHS of (b) takes (I) to (II)

— Similarly, re-express 3^{rd} rotation in (a):

$$R_{z'}(\gamma) = R_{y'}(\beta)\, R_z(\gamma)\, R_{y'}^{-1}(\beta) \quad \cdots (c)$$

Combing (a), (b) & (c)

$$R(\alpha, \beta, \gamma) = \underbrace{\left[R_{z'}(\gamma)\right]}_{(a)} R_{y'}(\beta)\, R_z(\alpha)$$


(I)
z
α
y'
α
y
x
α
x'
1st rotation
"-" α about z
(space/body)
z
(0)
y
β
x
2nd β about y
net β about y'
z
z'
α
β
(II)
y'
α
β
α
x''
z
α
z̃
β
y
3rd α about z
β
x̃

$$= \left[R_{y'}(\beta) R_z(\gamma) \cancel{R_{y'}^{-1}(\beta)} \right] \leftarrow (c)$$

$$\times \; \cancel{R_{y'}(\beta)} \; R_z(\alpha)$$

$$= \left[R_z(\alpha) \; R_y(\beta) \; \cancel{R_z^{-1}(\alpha)} \right] \leftarrow (b)$$

$$\times \; R_z(\gamma) \; \cancel{R_z(\alpha)}$$

[Rotations about same axis (z here) do commute]

$$= \; R_z(\alpha) \; R_y(\beta) \; R_z(\gamma)$$

(as promised): use above form in QM below

$\boxed{\text{Note}}$: in "new" $R(\alpha, \beta, \gamma)$, (α, β, γ) appear in opposite order to old $R(\alpha, \beta, \gamma)$ ... & all rotations about space axes

Euler angles: applied to $\boxed{\text{spin-}\tfrac{1}{2}}$ in QM
(higher angular momentum later)

- R (classical) $\longrightarrow$ $\mathcal{D}$ (QM operator in ket space)

- For spin-½ in 2-component:

$$\mathcal{D}(\hat{n},\phi) \doteq \Sigma \equiv \exp\left(\frac{-i\vec{\sigma}\cdot\hat{n}\phi}{2}\right)$$

so that $\mathcal{D}(\alpha,\beta,\gamma) = \mathcal{D}_z(\alpha)\mathcal{D}_y(\beta)\mathcal{D}_z(\gamma)$

$$\doteq \underbrace{\exp\left(\frac{-i\sigma_3\alpha}{2}\right)}_{\text{diagonal, complex}} \underbrace{\exp\left(\frac{-i\sigma_2\beta}{2}\right)}_{\boxed{\text{off-diagonal but real}}} \underbrace{\exp\left(\frac{-i\sigma_3\gamma}{2}\right)}_{\text{diagonal, complex}}$$

$$= \begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{+i\alpha/2} \end{pmatrix} \begin{pmatrix} \cos\beta/2 & -\sin\beta/2 \\ \sin\beta/2 & \cos\beta/2 \end{pmatrix} \begin{pmatrix} e^{-i\gamma/2} & 0 \\ 0 & e^{i\gamma/2} \end{pmatrix}$$

$$= \begin{pmatrix} e^{-i(\alpha+\gamma)/2}\cos\beta/2 & -e^{-i(\alpha-\gamma)/2}\sin\beta/2 \\ e^{-i(\alpha-\gamma)/2}\sin\beta/2 & e^{+i(\alpha+\gamma)/2}\cos\beta/2 \end{pmatrix}$$

Most general SU(2) matrix in Euler angle form

- Note: only off-diagonal matrix (in middle) is real,
cf. if middle/2nd rotation in (a) is about x'-axis (a la Goldstein), then middle would be $\boxed{\text{complex}}$ & $\boxed{\text{off}}$ diagonal ("inconvenient")
- this is $j = 1/2$ irreducible representation of $\mathcal{D}(\alpha, \beta, \gamma)$:

$$\mathcal{D}^{1/2\,(=j)}_{\underbrace{m'}_{1,2}\ \underbrace{m}_{1,2}} = \langle j = 1/2, m' | \exp\left(-\frac{iJ_z}{\hbar}\alpha\right) \exp\left(-\frac{iJ_y\beta}{\hbar}\right) \times \exp\left(-\frac{iJ_z\gamma}{\hbar}\right) | j = 1/2, m \rangle$$

Eigenvalues & Eigenstates of $\bar{J}$

- spin-$\frac{1}{2}$ is smallest realization ($N = 2$) of $[J_i, J_K] = i\varepsilon_{ijk} J_K$
- Onto higher (orbital or spin)...

... goal is (but long road to it):

$$\boxed{\begin{aligned} |\bar{J}|^2 |j, m\rangle &= j(j+1)\hbar^2 |j, m\rangle \\ J_z |j, m\rangle &= m\hbar |j, m\rangle \end{aligned}}$$

where $|\bar{J}|^2 = \sum_{i=1,2,3} J_i^2$; $j =$ integer or half-integer

and $m = (-j), (-j+1) \ldots (j-1), (+j)$ (degeneracy)

Also, raising/lowering operators:

$\boxed{J_\pm = J_x \pm i J_y}$ give

$$\boxed{J_\pm |j, m\rangle = \sqrt{(j \mp m)(j \pm m + 1)}\, \hbar\, |j, m \pm 1\rangle}$$

... seen for spin-$\frac{1}{2}$ ($\bar{J} \rightarrow \bar{S}$) already

Proof (buckle up!) From

$[J_i, J_j] = i\varepsilon_{ijk} J_k \hbar$ ... (1)

[based on J_i generates infinitesimal rotations + properties of rotations (not using $\bar{L} = \bar{x} \times \bar{p}$ (orbital angular momentum) ... so applies to spin (instrinsic angular momentum) also]

- Check $\left[|\bar{J}|^2, J_k \right] = 0$ $(k = 1, 2, 3)$ ...(2)

where $|\bar{J}|^2 \equiv J_x^2 + J_y^2 + J_z^2$

$\Rightarrow$ choose J_z to be simultaneously diagonalized with $|\bar{J}|^2$

$$|\bar{J}|^2 |a, b\rangle = a \, |a, b\rangle$$

$\hookrightarrow |\bar{J}|^2$ eigenvalue

$$\& \; J_z |a, b\rangle = b \, |a, b\rangle \quad \ldots (3)$$

$\hookrightarrow J_z$ eigenvalue

$|a, b\rangle$ eigenkets of $J_{x,y}$? No

- Next, lowering/raising (ladder) or annihilation/creation operators:

$$\boxed{J_{\pm} \equiv J_x \pm i J_y}$$

$$[J_+, J_-] = 2\hbar J_z \; ; \; [J_z, J_{\pm}] = \pm\hbar J_{\pm}$$

and $[|\vec{J}|^2, J_{\pm}] = 0 \quad \ldots (4)$