Lecture $\boxed{21}$, Oct. 19 (Mon.)

Outline for today

- Rotations of spin-$1/2$ using Pauli 2-component formalism

- More group theory of rotations: SO(3) formed by R's (3 x 3 orthogonal matrices) vs. SU(2) formed by spin-$1/2$ rotations

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cf. "abstract" thus far

Recall $\boxed{\text{matrix}}$ representation for spin-$1/2$ operators (and eigenkets):

$$|+\rangle \doteq \begin{pmatrix} 1 \\ 0 \end{pmatrix} \equiv \chi_+ \ \& \ |-\rangle \doteq \begin{pmatrix} 0 \\ 1 \end{pmatrix} \equiv \chi_- \text{ are}$$

represented by...

eigenkets (eigenvalues $\pm \hbar/2$) of $S_z \doteq \frac{\hbar}{2}\sigma_3$,

with $S_{x,y} \doteq \hbar/2 \, \sigma_{1,2}$, where Pauli matrices

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \; ; \; \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \; \& \; \sigma_2 = \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix}$$

[e.g. $S_x = \hbar/2 \{ |+\rangle\langle -| + |-\rangle\langle +| \}$ gives

$\langle \pm | S_x | \mp \rangle = 1$ & $\langle \pm | S_x | \pm \rangle = 0$, which

are matrix elements of S_x]

— So, general ket $|\alpha\rangle = |+\rangle\langle +|\alpha\rangle + |-\rangle\langle -|\alpha\rangle$

$$\doteq \begin{pmatrix} c_+ \equiv \langle +|\alpha\rangle \\ c_- \equiv \langle -|\alpha\rangle \end{pmatrix} \equiv \chi \text{ (spinor)}$$

— And, $\langle S_k \rangle \equiv \langle \alpha | S_k | \alpha \rangle$

$$= \sum_{a'=+,-} \sum_{a''=+,-} \langle \alpha | a' \rangle \overbrace{\langle a' | S_k | a'' \rangle}^{\hbar/2\, \sigma_k} \langle a'' | \alpha \rangle$$

$$= \boxed{\hbar/2 \; \chi^\dagger \sigma_k \chi} \quad \text{(matrix multiplication)}$$

— Properties of Pauli matrices:

$\sigma_i^2 = \mathbb{1}_{2\times 2}$; $\{\sigma_i, \sigma_j\} \equiv \sigma_i \sigma_j + \sigma_j \sigma_i = 2\delta_{ij}$;

$[\sigma_i, \sigma_j] = 2i\, \varepsilon_{ijk} \sigma_k$ (combining $[\dots]$ & $\{\dots\}$ $\;\sigma_1\sigma_2 = -\sigma_2\sigma_1 = i\sigma_3$);

$\sigma_i^\dagger = \sigma_i$; $\det(\sigma_i) = -1$; $\mathrm{tr}\, \sigma_i = 0$

– Useful to study : $\bar{\sigma}\cdot\bar{a}$ (2×2 matrix)

3d vector

$$= \begin{pmatrix} a_3 & a_1 - i a_2 \\ a_1 + i a_2 & -a_3 \end{pmatrix}$$

with $(\bar{\sigma}\cdot\bar{a})(\bar{\sigma}\cdot\bar{b}) = \bar{a}\cdot\bar{b} + i\bar{\sigma}\cdot(\bar{a}\times\bar{b})$, so $\bar{a}$ real gives $\boxed{(\bar{\sigma}\cdot\bar{a})^2 = |\bar{a}|^2}$

$\Rightarrow \mathcal{D}(\hat{n},\phi) = \exp\left(-i\frac{\bar{S}\cdot\hat{n}}{\hbar}\phi\right)$ acting on $|\alpha\rangle$

represented by

$\boxed{\chi \text{ (spinor)} \longrightarrow \Sigma\, \chi}$, where

$$\boxed{\Sigma \text{ (2×2 complex matrix)} = \exp\left(\frac{-i\bar{\sigma}\cdot\hat{n}\,\phi}{2}\right)}$$

– Using $(\bar{\sigma}\cdot\bar{a})^2 = |\bar{a}|^2$ with $\bar{a} = \hat{n}$ gives

$$(\sigma\cdot\hat{n})^m = \begin{cases} \mathbb{1} & \text{even } m \\ \sigma\cdot\hat{n} & \text{odd } m \end{cases}$$

so that

$$\boxed{\Sigma = \mathbb{1}\cos\phi/2 - i\,\bar{\sigma}\cdot\hat{n}\sin\phi/2}$$

$$= \begin{bmatrix} \cos\phi/2 - i n_z \sin\phi/2 & (-i n_x - n_y)\sin\phi/2 \\ (-i n_x + n_y)\sin\phi/2 & \cos\phi/2 + i n_z \sin\phi/2 \end{bmatrix}$$

– Σ is unitary: $\Sigma^\dagger \Sigma = \mathbb{1}$

"length" of $\chi = \sqrt{|c_+|^2 + |c_-|^2}$ preserved

– Note $\sigma_{k=1,2,3}$ do not rotate: instead $\boxed{\chi^\dagger \sigma_k \chi}$ does

$$\boxed{\chi^\dagger \sigma_k \chi \longrightarrow \sum R_{k\ell}\, \chi^\dagger \sigma_k \chi}$$

[expected: $\chi^\dagger \sigma_k \chi = \langle S_k \rangle$, which we earlier showed rotates]

– check explicitly: "new" $\chi^\dagger \sigma_k \chi =$

$$\chi^\dagger \exp\left(+\frac{i\bar{\sigma}\cdot\hat{n}}{2}\phi\right) \sigma_k \exp\left(-i\bar{\sigma}\cdot\hat{n}\,\phi/2\right)\chi :$$

– show $\exp(i\sigma_3\phi/2)\,\sigma_1 \exp(-i\sigma_3\phi/2)$

$\hat{n} = \hat{z}$ $\qquad$ $1 = k$

$$= \sigma_1 \cos\phi - \sigma_2 \sin\phi$$

[as expected from earlier general result:

$$\exp(iS_z\phi/\hbar)\, S_x \exp\left(\frac{-iS_z\phi}{\hbar}\right) = S_x\cos\phi - S_y \sin\phi \,]$$

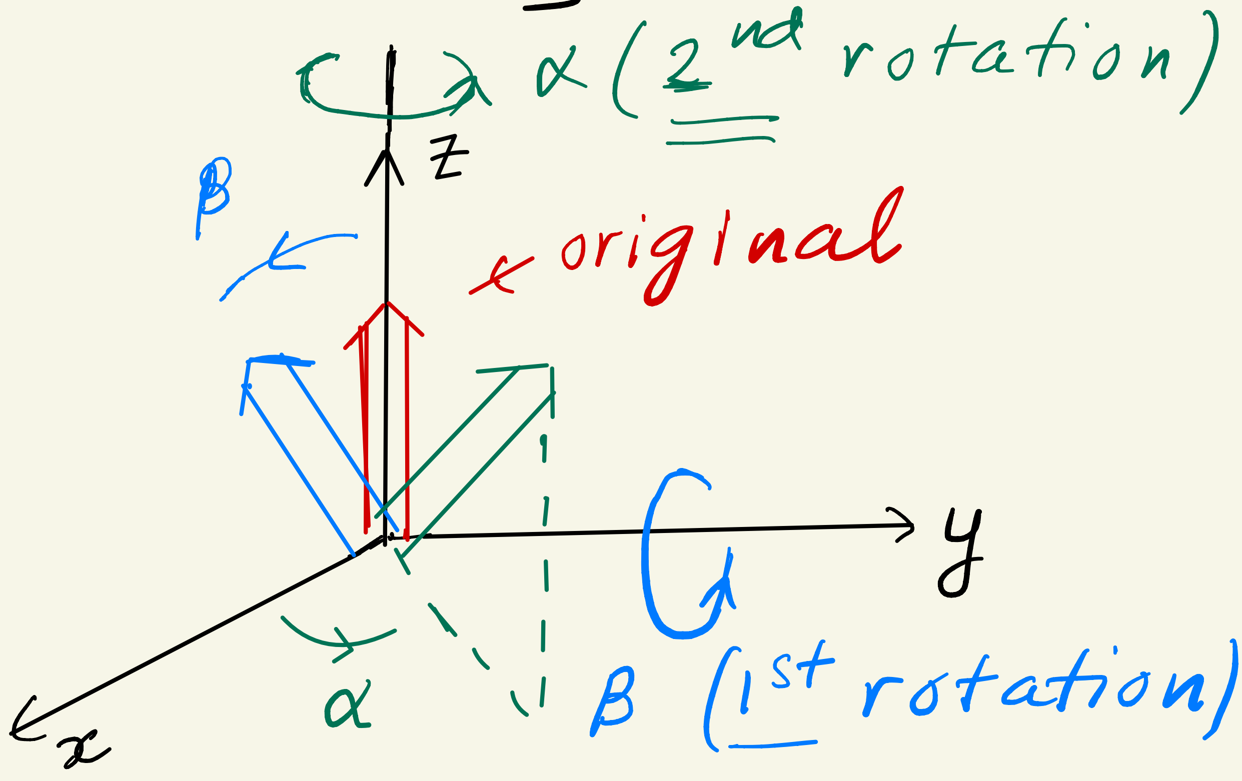
— Ket flipping sign under 2π rotation due here (matrix representation)

$$\boxed{\exp\left(-i\,\frac{\vec{\sigma}\cdot\hat{n}}{2}\,(\phi=2\pi)\right) = -\mathbb{1}}$$ for any $\hat{n}$

$$\left(= \mathbb{1}\cos 2\pi/2 - i\,\vec{\sigma}\cdot\hat{n}\,\sin\frac{2\pi}{2}\right)$$

— $\boxed{\text{Alternate}}$ way to solve for eigenspinor of $\boxed{(\vec{\sigma}\cdot\hat{n})\frac{\hbar}{2} \equiv S_n}$ with eigenvalue $+1$ (i.e., spin-$1/2$ component in general direction)

[see HW $\boxed{2.1}$ for brute force, i.e., diagonalization ...]



$$\underbrace{\exp\left(-i\sigma_3 \frac{\alpha}{2}\right)}_{\substack{2^{nd}\text{ rotation}\\ (z\text{-axis, }\alpha)}} \; \underbrace{\exp\left(-i\sigma_2 \beta/2\right)}_{\substack{1^{st}\text{ rotation}\\ (y\text{-axis, }\beta)}} \; \underbrace{\begin{pmatrix}1\\0\end{pmatrix}}_{\substack{\text{start with}\\ |+\rangle \text{ or } \chi_+}} =$$

$$= \left(\mathbb{1}\cos\alpha/2 - i\sigma_3 \sin\frac{\alpha}{2}\right)\left(\cos\frac{\beta}{2}\,\mathbb{1} - i\sigma_2 \sin\frac{\beta}{2}\right)\begin{pmatrix}1\\0\end{pmatrix}$$

$$= \begin{pmatrix}\cos\beta/2 \; e^{-i\alpha/2} \\ \sin\beta/2 \; e^{+i\alpha/2}\end{pmatrix} \text{ is eigenspinor of } S_n$$

(agrees with earlier way)

- So far, 2 ways to implement rotations: R (3 x 3 real orthogonal matrices) & $\Sigma = \exp\left(-\frac{i\vec{\sigma}\cdot\hat{n}\phi}{2}\right)$

- Next, more precise math to relate these 2 ...

- $\boxed{R\text{'s}}$ form $\boxed{\text{group}}$ called

$SO(3)$

S → special: determinant = 1

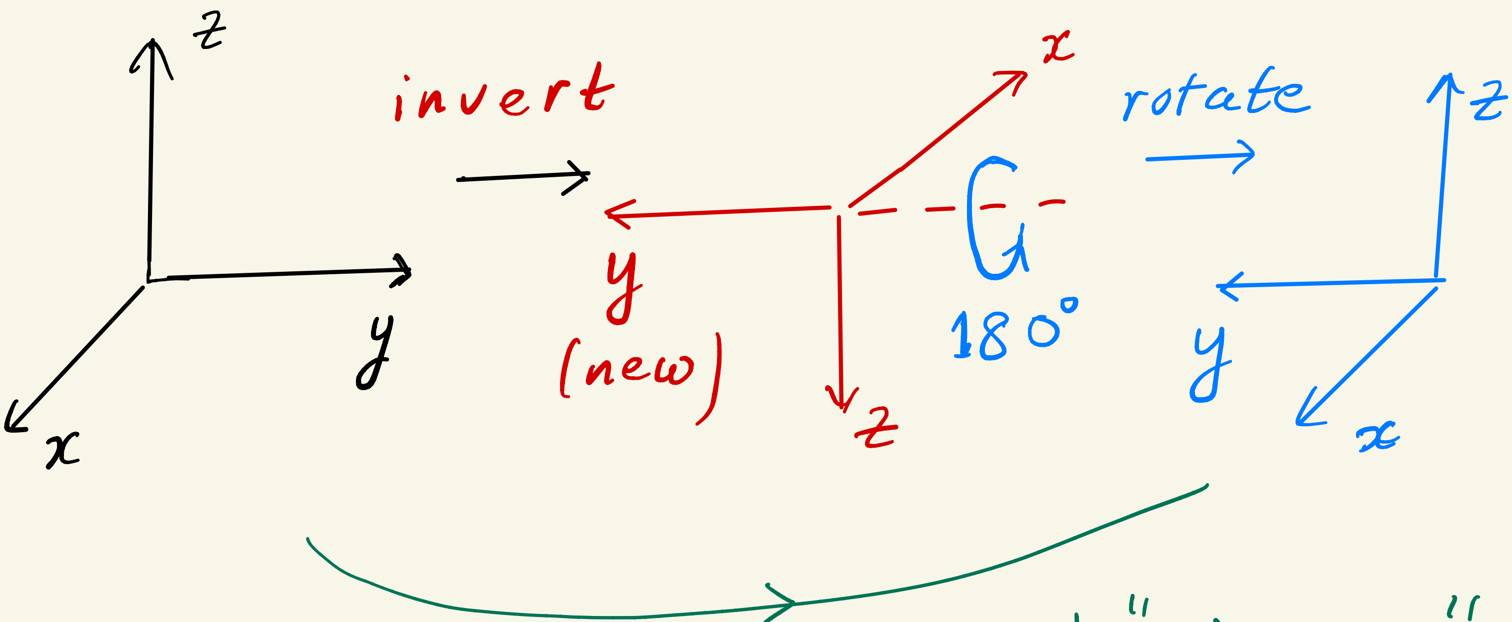
O → orthogonal

3 → dimension of matrix

vs. $O(3)$ includes inversion (keeps vector length unchanged, but (determinant = −1)

$R_{\text{inversion}} = \text{diag}(-1,-1,-1)$ (not really rotation)

— Reflection equivalent to inversion, followed by rotation



reflect in (z − x) "mirror"

Number of independent parameters in R =

9 (general real matrix) −

6 (orthogonality constraint: $\underbrace{R^T R}_{\text{symmetric (any } R)} = \mathbb{1}$)

3 diagonal

+ 3 off-diagonal

independent entries

= $\boxed{3}$

— Sanity check : simpler parametrization of rotation by polar & azimuthal angle of axis of rotation (unit vector $\hat{n}$) and angle of rotation ϕ : $\boxed{3}$ parameters (x, y, z components of $\hat{n}\phi$) ...

... but $\left(\hat{n}_1 \phi_1 + \hat{n}_2 \phi_2\right)$ does not correspond to $\boxed{\text{successive}}$ rotations, e.g., $\hat{n}_{1,2} = \hat{y}, \hat{x}$, $\phi_{1,2} = 90^\circ \Rightarrow$

$\boxed{\text{sum}}$ of vectors $= \underbrace{\left(\sqrt{2} \times 90^\circ\right)}_{\simeq 130^\circ} \underbrace{\left(\frac{\hat{x} + \hat{y}}{\sqrt{2}}\right)}_{45^\circ \text{ to } x \,\&\, y\text{-axes}}$

— So, R's (where $R_1 R_2$ does give result of successive rotations) is "better" parametrization