Lecture $\boxed{20}$, Oct. 16, 2020 (Fri.): $\boxed{\text{part (II)}}$

Outline

- spin-$1/2$ systems
  - connect earlier discussion to rotations & angular momentum
  - time-evolution of spin in $\bar{B}$ is "rotation" (use to detect flip of sign of ket under 2π rotation from spin-precession)
  - Pauli matrices, spinor language

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$\boxed{\text{How to rotate spin-}1/2}$

- $\boxed{N=2}$ smallest dimensionality of ket space where $[J_i, J_j] = i\varepsilon_{ijk} J_k$ works: $S_{x,y,z}$ of before satisfy $[S_i, S_j] = i\varepsilon_{ijk} S_k$

$$S_x = +\hbar/2 \left\{ |+\rangle\langle-| + |-\rangle\langle+| \right\}$$

$$S_y = i\hbar/2 \left\{ -|+\rangle\langle-| + |-\rangle\langle+| \right\}$$

$$S_z = +\hbar/2 \left\{ |+\rangle\langle+| - |-\rangle\langle-| \right\}$$

- what's "new"? ϕ rotation about z-axis:

$|\alpha\rangle_{R\,(\text{otated})} = \mathcal{D}_z(\phi)\underbrace{|\alpha\rangle}_{\text{old}}$ (in general)

with $\boxed{\mathcal{D}_z(\phi) = \exp(-iS_z\phi/\hbar) \text{ for spin-}1/2}$

- (Sanity) check: $\exp(-iS_z\phi/\hbar)$ does rotate based on effect on $\langle S_{x,y}\rangle$:

$\boxed{\text{expect}}$: $\underbrace{\langle S_x\rangle}_{\langle\alpha|S_x|\alpha\rangle} \rightarrow {}_R\langle\alpha|S_x|\alpha\rangle_R = \langle S_x\rangle\cos\phi - \langle S_y\rangle\sin\phi$

Proof: ${}_R\langle\alpha|S_x|\alpha\rangle_R = \langle\alpha|\mathcal{D}_z^\dagger(\phi)S_x\mathcal{D}_z(\phi)|\alpha\rangle$

work out $\boxed{\exp(iS_z\phi/\hbar)\,S_x\exp(-iS_z\phi/\hbar)}$

$\boxed{1^{st}}$ way (brute force: plug above S_x)

if $X|\alpha\rangle = c|\alpha\rangle$, then $\langle\alpha|X^\dagger = c^*\langle\alpha|$

$\exp(-i\phi/2)$ $\exp(+i\phi/2)$

$$\left(\frac{\hbar}{2}\right)\exp\left(\frac{iS_z\phi}{\hbar}\right)\left\{|+\rangle\langle-| + |-\rangle\langle+|\right\}\exp\left(-\frac{iS_z\phi}{\hbar}\right)$$

$\exp(+i\phi/2)$ $\exp(-i\phi/2)$

$$= \frac{\hbar}{2}\left(e^{i\phi/2}|+\rangle\langle-|e^{+i\phi/2} + e^{-i\phi/2}|-\rangle\langle+|e^{-i\phi/2}\right)$$

$$= \hbar/2\left(\{|+\rangle\langle-| + |-\rangle\langle+|\}\cos\phi + i\{|+\rangle\langle-| - |-\rangle\langle+|\}\sin\phi\right)$$

$$= (S_x \cos\phi - S_y \sin\phi)$$

—— x ——

$\boxed{2^{nd}}$ option (generalizes to higher angular momentum)

$\boxed{\text{Use}}$
$$\exp(i G\lambda)\, A\, \exp(-iG\lambda) = A + i\lambda[G,A] + \frac{i^2\lambda^2}{2!}[G,[G,A]] + \ldots$$

Here, $G = S_z$, $A = S_x$, $\lambda = \phi/\hbar$ ... so we get on RHS :

$$S_x + \frac{i\phi}{\hbar}\underbrace{[S_z, S_x]}_{i\hbar S_y} + \frac{1}{2!}\left(\frac{i\phi}{\hbar}\right)^2 \underbrace{[S_z, \underbrace{[S_z, S_x]}_{i\hbar S_y}]}_{\hbar^2 S_x} + \ldots$$

$$= S_x\left[1 - \phi^2/2! + \ldots\right] - S_y\left[\phi - \phi^3/3! + \ldots\right]$$

$$= S_x \cos\phi - S_y \sin\phi$$

Similarly, $\langle S_y\rangle \longrightarrow \langle S_y\rangle \cos\phi + \langle S_x\rangle \sin\phi$

$\Rightarrow \mathcal{D}_z(\phi)$ does rotate ket (as expected):

$$\langle S_k\rangle \longrightarrow \sum_l R_{kl}\langle S_l\rangle$$

R_{kl}: 3×3 orthogonal

... from $\boxed{2^{nd}}$ way (used $\boxed{\text{only}}$ commutation relations) : $\langle J_k\rangle \longrightarrow \sum_l R_{kl}\langle J_l\rangle$ ← any ...

... but we encountered $\boxed{\text{rotation}}$ of spin $\boxed{\text{earlier}}$!

spin - precession : time - evolution with $\bar{B}$

$$H = -\left(\frac{e}{m_e c}\right) \bar{S} \cdot \bar{B} = \omega S_z , \quad \omega = |e| B / (m_e c)$$

$$\Rightarrow \underbrace{U(t, 0)}_{\text{time-evolution operator}} = \exp\left(\frac{-iHt}{\hbar}\right) = \exp\left(\frac{-iS_z \omega t}{\hbar}\right)$$

... same as $\mathcal{D}_z(\phi)$, with $\boxed{\phi = \omega t}$ $\Rightarrow$

clear why U causes spin precession/rotation: earlier

$$\langle S_x \rangle_{\boxed{t}} = \langle S_x \rangle_{t=0} \cos(\underbrace{\omega t}_{\phi \text{ here}}) - \langle S_y \rangle \sin(\underbrace{\omega t}_{\phi})$$

(alternate solution to HW 4.2)

after $\boxed{t = 2\pi/\omega}$, spin returns to initial value

$$\left(\tau_{\text{precession}} = 2\pi/\omega\right)$$

— ket evolution : $|\alpha, t_0 = 0; t\rangle$

$$= e^{-i\omega t/2} |+\rangle\langle +|\alpha\rangle + e^{i\omega t/2} |-\rangle\langle -|\alpha\rangle$$

$\Rightarrow$ $\boxed{\tau_{\text{ket}}}$ $\equiv$ time for ket to get back to initial

$$= 4\pi/\omega = 2 \, \boxed{\tau_{\text{precession}}}$$

In terms of $\boxed{\text{rotations}}$ ("implemented" by time evolution above),

$$\mathcal{D}_z(\phi)|\alpha\rangle = e^{-i\phi/2}|+\rangle\langle+|\alpha\rangle + e^{+i\phi/2}|-\rangle\langle-|\alpha\rangle$$

Set $\boxed{\phi = 2\pi}$, $\boxed{|\alpha\rangle_R = -|\alpha\rangle}$ ... "need" ($\phi=$) 4π rotation to get back $|\alpha\rangle$

... but $\langle S_x\rangle$ only takes 2π, since $\langle\alpha|S_x|\alpha\rangle \longrightarrow (-1)^2\langle\alpha|S_x|\alpha\rangle$

$\boxed{\text{spin precession}}$ tested with muons (electric charge $-|e|$, but $m_\mu \simeq 200\, m_e$):

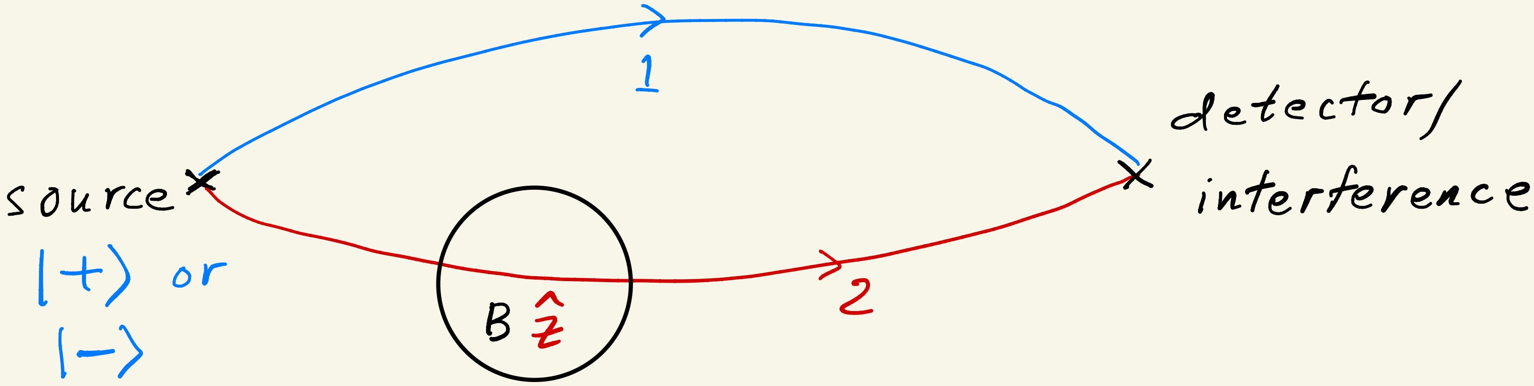
- μ_μ ← muon measured via hyperfine splitting of $\boxed{\mu^+ e^-}$ (muonium), like H-atom: electron spin-μ proton interaction: μ_μ (similar to μ_e: any elementary spin-½) $= e\hbar/(2m_\mu c)$

- $\boxed{\text{muon in } \vec{B}}$: precession measured by direction of electrons from muon $\boxed{\text{decay}}$ (electrons emitted in opposite direction to muon spin)

— How can we "rotate" ket by 2π in order to observe sign flip?

We need to compare / interfere rotated & unrotated state

— Use time evolution in $\bar{B}$ to "implement" rotation: neutron interferometry



$\Rightarrow$ beam 2 has phase change (relative to 1):
$\exp(-i\omega T/2)$ (T is time spent by 2 in $\bar{B}$)
$\omega = g_n e B/(m_n c)$

— $\mu_{e \text{ or } \mu}$ (elementary charged spin-½ particle)
$= e\hbar/(m_{e \text{ or } \mu} c)\ 1/2$ ("due to current from spinning")

vs. $\mu_{\text{neutron}} = g_n e\hbar/2m_n c$, $g_n = -1.9$:
neutral, but composite (made of charged quarks), so still get (non-trivial) μ

- beam 1 has amplitude $c_1(B=0)$ at detector
- beam 2 : $\underbrace{c_2(B=0)}_{\text{set } c_1(B=0)} e^{(-i\omega T/2)}$

$\Rightarrow$ intensity at detector $\propto \boxed{\cos\left(-\frac{\omega T}{2}\right)}$

$\boxed{T}$ fixed, but ω varied using $B \Rightarrow$ $\boxed{\text{intensity}}$ sinusoidal in $\boxed{B}$:

ΔB for consecutive maxima:

path length

$\frac{T}{2}\Delta\omega = 2\pi$: use $T = \ell/v = \frac{\ell m_n}{p} = \frac{\ell m_n}{\hbar \lambda}$

momentum

de Broglie wavelength

$$\Rightarrow \boxed{\Delta B = \frac{4\pi\hbar c}{e g_n \lambda \ell}}$$

vs. $\Delta B = 2\pi\hbar c/(e g_n \lambda \ell)$ $\boxed{\text{if}}$ ket back to original with 2π rotation (naive expectation)