Lecture $\boxed{19}$, Oct. 14 (Wed.) & Lecture $\boxed{20}$, Oct. 16 (Fri.) part I

Outline for today:

- (magnetic) monopole: beautiful "combination" of non-trivial $\bar{A}$ (gauge transformation) and QM ... giving observable effect
  - $\bar{A}$ due to monopole (classical)
  - enter QM: (if monopole exists, then) magnetic charge is quantized
- start discussion of $\boxed{\text{angular momentum } (\bar{J})}$ relation to rotations (aka translations & linear momentum, $\bar{p}$)

$\boxed{\text{Maxwell's}}$ equations [$\bar{B}$ due to electric current or magnetic dipole] with monopole

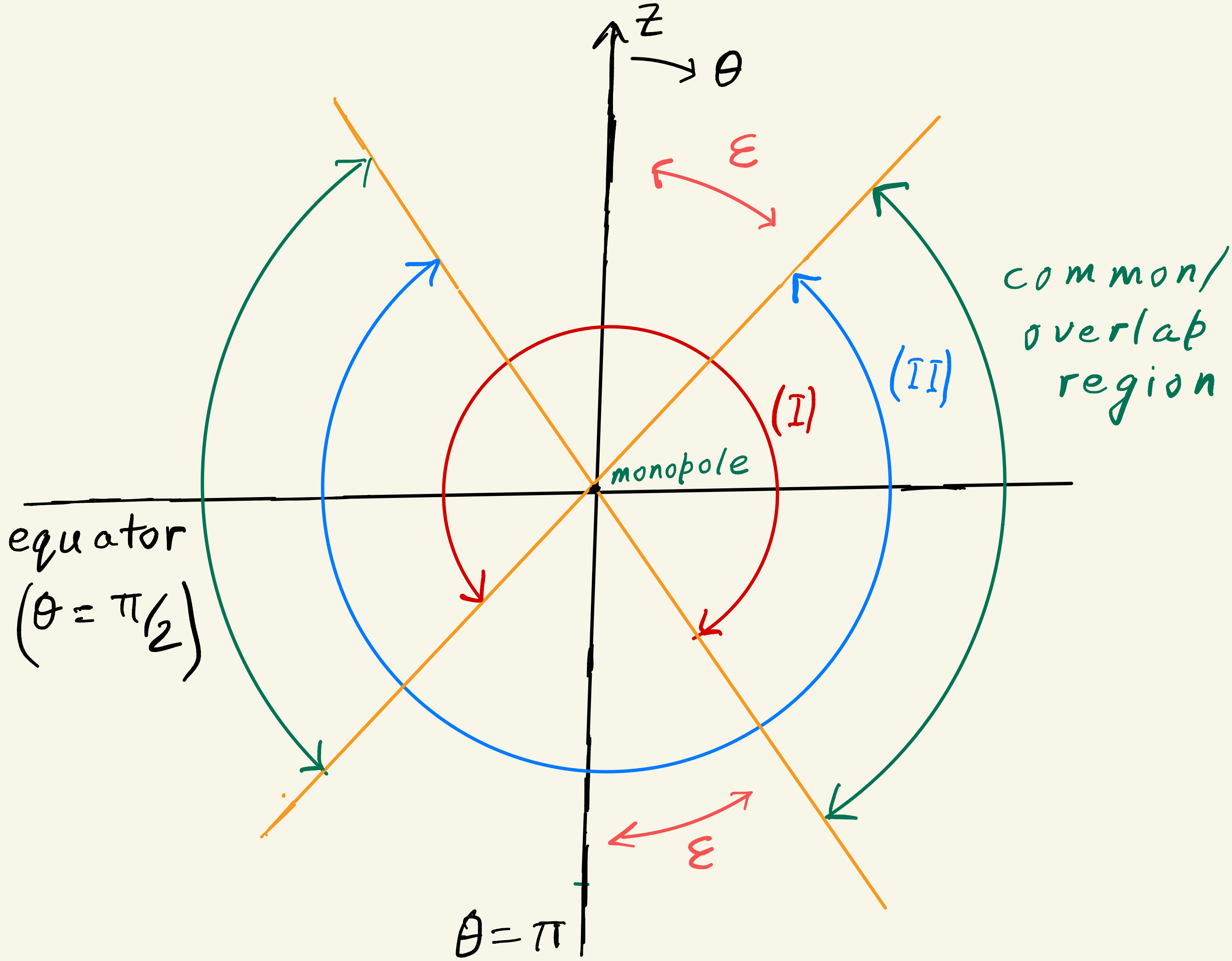
$$\bar{\nabla}\cdot\bar{E} = 4\pi\rho_E \qquad \boxed{\bar{\nabla}\cdot\bar{B} = 4\pi\rho_M}$$

$$\boxed{\bar{\nabla}\times\bar{E} = -\partial\bar{B}/\partial t + \frac{4\pi}{c}J_M} \qquad \bar{\nabla}\times\bar{B} = +\frac{1}{c}\frac{\partial\bar{E}}{\partial t} + \frac{4\pi}{c}J_E$$

Classically:

- Point magnetic charge (e_M) at origin gives $\boxed{\bar{B} = e_M\,\hat{r}/r^2}$ [a la $\bar{E}$ due to point electric charge: $\bar{\nabla}\cdot B$ (or $\bar{E}$) $= 4\pi\delta(\bar{r})\,e_M$ (or E)]

- Above $\bar{B}$ derivable from $\bar{A} = \bar{B} = \bar{\nabla} \times \bar{A}$?
- Recall: $\bar{B} = \bar{\nabla} \times \bar{A}$ based on $\bar{\nabla} \cdot B = 0$ ... but now $\bar{\nabla} \cdot \bar{B} = 4\pi\, \delta(r)\, e_M \neq 0$
- Expect no non-singular $\bar{A}$ valid at all points ... but do "patchwork", exploiting gauge transformation (two different A's can give same $\bar{B}$)



$$\boxed{\bar{A} = \left[\frac{e_M(1-\cos\theta)}{r\sin\theta}\right]\hat{\phi}}$$ gives above

$\bar{B}$ ... $\boxed{\text{but}}$ on $z<0$ axis $(\theta=\pi)$, it's singular $\left(\propto \frac{1+1}{0}\right)$ [on $z>0$ axis $(\theta=0)$, it's smooth $1-\cos\theta = 2\sin^2\theta/_2 \sim O(\theta^2)$, vs. $\sin\theta \sim \theta^1$ ...]

- Ambiguity in $\bar{A}$ (gauge transformation) helps: "patchwork"

$$A^{(I)} = \frac{e_M(1-\cos\theta)}{r\sin\theta}\hat{\phi} \qquad \boxed{\theta < \pi - \varepsilon}$$

$$A^{(II)} = -\frac{e_M(1+\cos\theta)}{r\sin}\hat{\phi} \qquad \boxed{\theta > \varepsilon}$$

Overlap region: both A's give same $\bar{B}$ $\Rightarrow$ related by gauge transformation:

$$\left(A^{(II)} - A^{(I)}\right) = -\frac{2e_M}{r\sin\theta}\hat{\phi} = \bar{\nabla}\Lambda, \quad \boxed{\Lambda = -2e_M\phi}$$

Onto $\boxed{\text{QM}}$: 2 ψ's of charged particle related:

$$\psi^{(II)} = \psi^{(I)} \exp\Big(\underbrace{-\frac{2iee_M\phi}{\hbar c}}_{ie\Lambda/\hbar c}\Big)$$

– each $\boxed{\psi}$ single-valued: for fixed r, θ ("given latitude"), same point

$$\psi^{(I)}(\phi=0) = \psi^{(I)}(\phi=2\pi) \quad \&$$

$$\underbrace{\psi^{(II)}(\phi=0)}_{\psi^{(I)}(\phi=0)\times\exp\left(-\frac{2iee_M 0}{\hbar c}\right)} = \underbrace{\psi^{(II)}(\phi=2\pi)}_{\psi^{(I)}(\phi=2\pi)\exp\left(-\frac{2iee_M 2\pi}{\hbar c}\right)}$$

$\left.\exp\left(-\frac{2iee_M 0}{\hbar c}\right)\right\} = 1$

$\Rightarrow$ (combining) $\exp\left[-2iee_M 2\pi/(\hbar c)\right] = 1 \Rightarrow$

$$-2ee_M 2\pi/(\hbar c) = 2\pi N$$

$$\boxed{e_M = \frac{\hbar c}{2|e|}(\pm N) \quad N = 0, 1, 2 \ldots}$$

$\Rightarrow$ magnetic charge is quantized

Rotations / Angular momentum (chapter 3)

- So far, QM in 1 d ("kinematics": chapter 1 & dynamics: chapter 2): spatial translation of state ket generated by momentum operator (time evolution by Hamiltonian operator)
- Onto 3 d: (trivial extension) translation (& $\vec{p}$) in each dimension, e.g., 3 d free particle just "product" of three 1 d ...
- Next: more complicated "operation" in 3 d: Rotations generated by angular momentum operator (return to free 3 d particle with this idea)
- develop (initially) angular momentum theory in "analogy" with translations ... but crucial difference: translations along different directions commute, while rotations do not
- Two kinds of angular momentum ($\vec{J}$):
  - orbital ($\vec{L}$): "integer-valued" vs.
  - intrinsic/spin ($\vec{S}$): half or whole integer

(spin-½ done before, but did not "connect" to angular momentum)

Angular momentum (operator) commutation relations $\boxed{\text{from}}$ properties of rotations [like those of p from translations, $T(dx')$]

$\boxed{\text{Goal}}$: $[J_i, J_j] = \underbrace{\varepsilon_{ijk}}_{1,2,3} i\hbar J_k$

imaginary

How to represent rotations (in 3d)

by $\boxed{3 \times 3}$ matrices:

$$\underbrace{\begin{pmatrix} V_{x'} \\ V_{y'} \\ V_{z'} \end{pmatrix}}_{\text{components of rotated vector}} = \underset{\text{matrix}}{(R)}_{3\times3} \underbrace{\begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix}}_{\text{old vector}}$$

Condition: length of V is unchanged $\Rightarrow$ R is orthogonal:

$$\boxed{R^T R = \mathbb{1}}$$

[coordinate axes fixed...]

- Example rotation about z-axis by ϕ:

$$R_z(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

infinitesimally (like translation/time evolution)

$$\simeq \begin{pmatrix} \underbrace{1 - \varepsilon^2/2}_{\cos\cdots} & -\varepsilon & 0 \\ +\varepsilon & 1 - \varepsilon^2/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Similarly,

$$R_x \approx \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-\varepsilon^2/2 & -\varepsilon \\ 0 & \varepsilon & 1-\frac{\varepsilon^2}{2} \end{pmatrix}$$

$$R_y \approx \begin{pmatrix} 1-\varepsilon^2/2 & 0 & \varepsilon \\ 0 & 1 & 0 \\ -\varepsilon & 0 & 1-\varepsilon^2/2 \end{pmatrix}$$

$\Rightarrow$ rotation about y-axis, followed by x-axis given by $\boxed{R_{\underset{2^{nd}}{x}}(\varepsilon)\, R_{\underset{1^{st}}{y}}(\varepsilon)}$

vs. other order: $\boxed{R_y R_x}$

— do commute at $O(\varepsilon^1)$, but not at $O(\varepsilon^2)$ (will determine $[J_i, J_j]$)

$$R_x(\varepsilon) R_y(\varepsilon) - R_y(\varepsilon) R_x(\varepsilon) = \begin{pmatrix} 0 & -\varepsilon^2 & 0 \\ \varepsilon^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq 0$$

$$= R_z(\varepsilon^2) - \underbrace{\mathbb{1}}_{R_{any}(0)}$$

$$\boxed{R_x(\varepsilon)\cdot R_y(\varepsilon) - R_y(\varepsilon)\cdot R_x(\varepsilon) = R_z(\varepsilon^2) - R_{any}(0)}$$

Onto $\boxed{QM}$: just like translations:

$$|\alpha\rangle \longrightarrow \underbrace{T(dx')\,|\alpha\rangle}_{\text{new (shifted by } dx')}$$

$$\boxed{\underbrace{|\alpha\rangle_{R(\text{otated})}}_{\text{new}} = \underbrace{\mathcal{D}(R)}_{\text{operator}}\ \underbrace{|\alpha\rangle}_{\text{old}}}$$

operator [corresponding to (3×3) orthogonal matrix R]

– $\boxed{\mathcal{D}}$ acts on (state) vectors in ket space [vs. R on 3-column classical vectors in real space]

— matrix representation of $\mathcal{D}$: depends on dimensionality N of ket space ($N = 3$ for angular momentum $J = 1$; $N = 2$ for spin-$\frac{1}{2}$: $\mathcal{D}$ is 2×2 matrix)

— x —

Like for translations/time-evolution, "build" $\mathcal{D}$ infinitesimally:

$$U_\varepsilon \text{ (unitary)} \simeq \mathbb{1} - i G \varepsilon$$

Hermitian

translation: $G_x = p_x/\hbar$; $\varepsilon \Rightarrow dx'$

time-evolution: $G = H/\hbar$; $\varepsilon = dt'$

— Classically, generator of (infinitesimal) canonical rotation (transformation) is angular momentum (orbital: $\bar{x} \times \bar{p}$) (Goldstein: Eq. 9.114 on p. 404)

$\Rightarrow$ define $\boxed{J_k}$ (angular momentum operator): $G_k = J_k/\hbar$ $(k = 1, 2, 3)$

$\varepsilon = d\phi \Rightarrow \mathcal{D}(\hat{n}, d\phi) = \left[1 - i\, d\phi \left(\frac{\vec{J} \cdot \hat{n}}{\hbar}\right)\right]$

axis of rotation

- J is **not** defined as "$\bar{x} \times \bar{p}$"

... so, covers case of spin

- finite rotation: $\mathcal{D}_z(\phi) = \lim_{N \to \infty} \left[1 - \frac{iJ_z}{\hbar}\left(\frac{\phi}{N}\right)\right]^N = \exp\left(\frac{-iJ_z\phi}{\hbar}\right)$

Having defined J_i as above, get commutation relations using properties of rotations:

- every R (3 x 3 orthogonal matrix) $\longrightarrow \mathcal{D}(R)$ (operator)

$\Rightarrow$ properties of R inherited by $\mathcal{D}$

- set of R's constitute "group"

Detour on $\boxed{\text{group}}$ theory

Set of elements $(x, y \ldots)$, which can be "multiplied": $x \circ y$, satisfying

(1). identity element : $x \circ \mathbb{1} = x$
$(= \mathbb{1} \circ x)$

(2) inverse : $x^{-1} \circ x = x \circ x^{-1}$
$= \mathbb{1}$

(3). closure : $x \circ y = z$ (another element)

(4). associativity : $x \circ (y \circ z)$
$= (x \circ y) \circ z \equiv x \circ y \circ z$

e.g. (all) integers under addition ("$\mathbb{1}$"$=0$), but not multiplication; real numbers (except 0) under multiplication

— R's form group, e.g., closure :

$(R_1 R_2)^T (R_1 R_2) = R_2^T \underbrace{R_1^T R_1}_{\mathbb{1}} R_2 = R_2^T R_2 = \mathbb{1} \Rightarrow R_1 R_2$ is orthogonal

... $R^{-1} = R^T$ etc.

$\Rightarrow \mathcal{D}(R)$'s also form group

– So, $R_x(\varepsilon)R_y(\varepsilon) - R_y(\varepsilon)R_x(\varepsilon)$ ✓ $\mathbb{1}$

$$= R_z(\varepsilon^2) - R_{any}(0)$$

gives, with $R \longrightarrow \mathcal{D}$ and $\mathcal{D} \approx \left(\mathbb{1} - i\frac{\bar{J}\cdot\bar{n}\,d\phi}{\hbar}\right)$

$$\underbrace{\left(\mathbb{1} - i\frac{J_x}{\hbar}\varepsilon - \frac{J_x^2}{2\hbar^2}\varepsilon^2 + \dots\right)}_{\mathcal{D}_x(\varepsilon)}\underbrace{\left(\mathbb{1} - i\frac{J_y}{\hbar}\varepsilon - \frac{J_y^2}{2\hbar^2}\varepsilon^2 \dots\right)}_{\mathcal{D}_y(\varepsilon)}$$

– other way

$$= \underbrace{\left(\mathbb{1} - i\frac{J_z}{\hbar}\varepsilon^2\right)}_{\mathcal{D}_z(\varepsilon^2)} - \mathbb{1}$$

$\mathbb{1}$'s cancel

$O(\varepsilon^1) \propto J^1$

also cancel

$$\varepsilon^2\left[\left(-\frac{J_x^2}{2\hbar^2} - \frac{J_y^2}{2\hbar^2} - \frac{J_xJ_y}{\hbar^2}\right) - \left(-\frac{J_x^2}{2\hbar^2} - \frac{J_y^2}{2\hbar^2} - \frac{J_yJ_x}{\hbar^2}\right)\right]$$

$$= \varepsilon^2(-iJ_z/\hbar) \Rightarrow [J_x, J_y] = +i\hbar J_z$$

In general $\boxed{[J_i, J_j] = i\varepsilon_{ijk}\hbar J_k}$