Lecture 18, Oct. 12 (Mon.)

Outline for today (and Wed.)

- gauge transformation in EM: classical and quantum
- Warm-up for subtle effects from $\vec{A}$ (magnetic vector potential) [$\vec{A}$ not constant in region with $\vec{B} = 0$ (where particle moves)]: bound-state AB effect
- original AB effect
- Magnetic monople
- Common thread in above 3 configurations: interplay of non-trivial $\vec{A}$ (gauge transformation etc.) - already present at classical level - with quantum ...
- common theme of electrostatic, gravitational potential examples (previous lecture) with AB effect, monopole (with EM potential): no force (classically), but still non trivial effects for $\hbar \neq 0$

Most general, time-dependent EM:

$$\bar{E} = -\bar{\nabla}\phi - \frac{1}{c}\frac{\partial \bar{A}}{\partial t} \quad \& \quad \bar{B} = \bar{\nabla}\times\bar{A}$$

unaffected by gauge transformation:

$$\phi \longrightarrow \phi - \frac{1}{c}\frac{\partial}{\partial t}\Lambda(x,t) \quad \& \quad A \longrightarrow A + \bar{\nabla}\Lambda(x,t)$$

Classical implications of gauge transformation

- (charged) particle trajectory independent of gauge used [$\Lambda(x)$ chosen], e.g.

$$\bar{B} = B\hat{z} \text{ "from" } \begin{cases} A_x = -By/2;\ A_y = \frac{Bx}{2} \\ A_z = 0 & \ldots (1) \\ A_x = -By;\ A_y = 0;\ A_z = 0 & \ldots (2) \end{cases}$$

(1) & (2) related by $\bar{A} \rightarrow \bar{A} - \bar{\nabla}\left(\frac{Bxy}{2}\right)$, $\Lambda(x)$

- trajectory [same for (1) & (2)]: $\boxed{\text{helix}}$ (uniform circular motion in x-y plane + uniform linear in z)

- p_x (similarly p_y) for 2 $\bar{A}$ choices are different:

$$dp_x/dt = -\partial H/\partial x \begin{cases} \neq 0 & \text{for } (1) \\ = 0 & \text{for } (2) \end{cases}$$

- $\boxed{\text{trajectory}}$ given by $\boxed{\pi = m\, d\bar{x}/dt}$

$$\pi = p - eA/c$$

gauge-invariant

gauge-dependent

Onto QM (expectation values behave classically) : $\langle x \rangle$ & $\langle \pi \rangle$ gauge-invariant

- $|\alpha\rangle$ for $\bar{A} \longrightarrow |\tilde{\alpha}\rangle$ for $\tilde{A} = A + \bar{\nabla}\Lambda$

operators (via x)

- require :

(1). $\langle \alpha | x | \alpha \rangle = \langle \tilde{\alpha} | x | \tilde{\alpha} \rangle$

(2). $\langle \alpha | \, p - \frac{eA}{c} | \alpha \rangle = \langle \tilde{\alpha} | \, p - \frac{e\tilde{A}}{c} | \tilde{\alpha} \rangle$

(3). $\langle \alpha | \alpha \rangle = \langle \tilde{\alpha} | \tilde{\alpha} \rangle$

If $|\tilde{\alpha}\rangle \equiv G\,|\alpha\rangle$, then "sufficient" condition:

(1). $\langle\tilde{\alpha}|\,x\,|\tilde{\alpha}\rangle = \langle\alpha|\,G^{+}x\,G\,|\alpha\rangle \Rightarrow$

$\boxed{G^{+}x\,G = x}$ (2) $G^{+}\left(p - e\tilde{A}/c\right)G = p - eA/c$

(3) $G^{+}G = 1$ (unitary)

— claim: $\boxed{G = \exp\left[ie\,\Lambda(x)/\hbar c\right]}$

[Proof: (1) & (3) clear; for (2), use $[p, F(x)] = -i\hbar\frac{\partial F}{\partial x}$ to compute $\exp\left(-\frac{ie\Lambda}{\hbar c}\right)p\,\exp\left(+\frac{ie\Lambda}{\hbar c}\right) = \cdots = p + \frac{e}{c}\nabla\Lambda$]

— Onto $\tilde{\psi}(x',t) = \exp\left[\frac{ie}{\hbar c}\Lambda(x')\right]\psi(x',t)$

— check from SE for $|\alpha\rangle$ or $\psi(x',t)$: if $|\alpha\rangle$ or $|\psi\rangle$ satisfies SE with A, then above $|\tilde{\alpha}\rangle$ or $\tilde{\psi}$ satisfies SE with $\boxed{A + \nabla\Lambda}$

— probabilities: density, $\rho = |\psi|^2$ gauge invariant

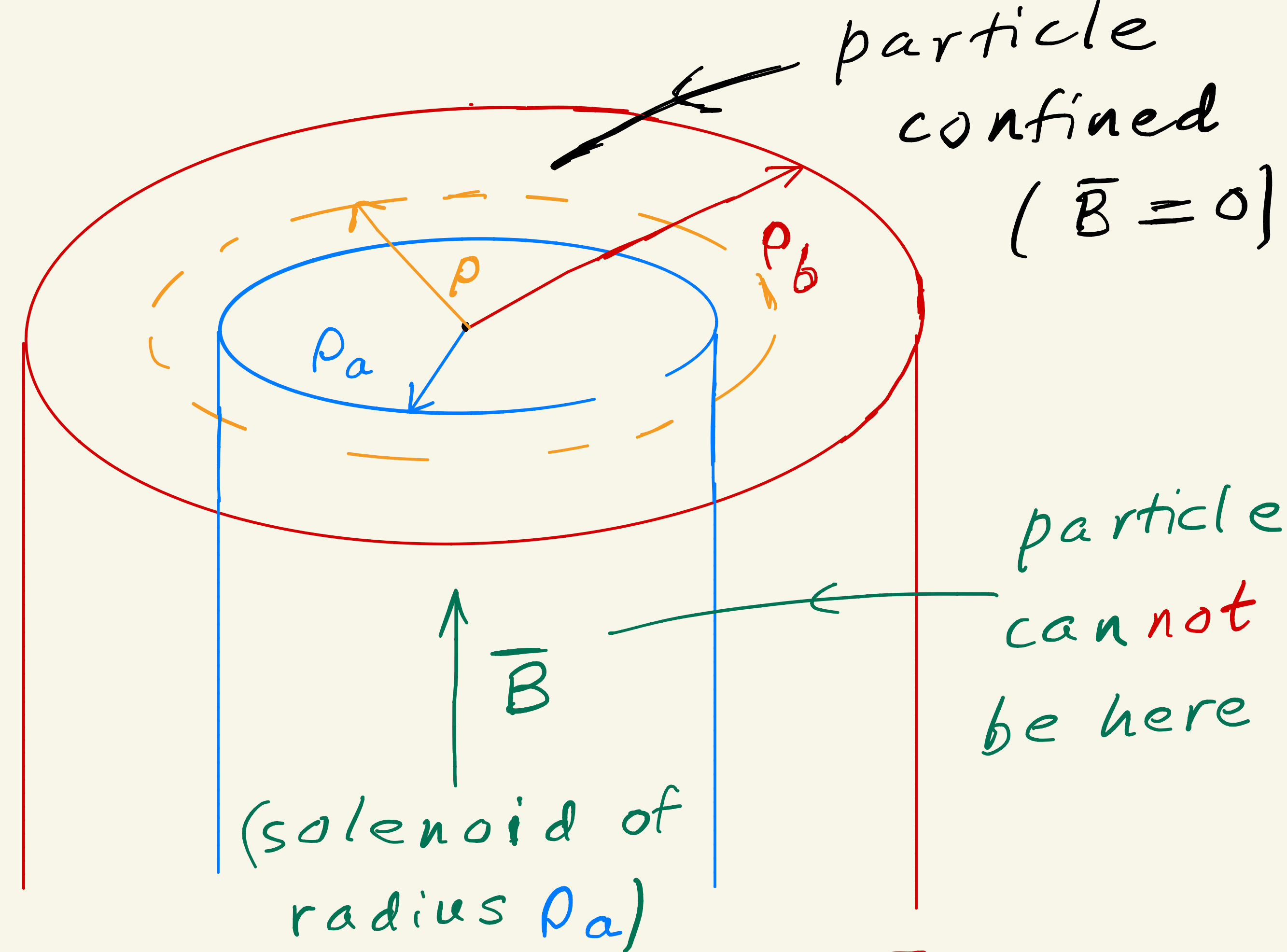
flux $j = \rho/m\left(\nabla S - \frac{e}{c}A\right)$ is also invariant

since S (phase of ψ) $\longrightarrow S + e\Lambda(x')/c$

— summary: state kets, p gauge-dependent

kinematical momentum, probability flux invariant

Bound-state AB effect



- BC : $\psi = 0$ at $\rho = \rho_a, \rho_b$ [independent of $\bar{B}$]

... naively, no effect of $\bar{B}$? $\boxed{\text{what about SE?}}$

- Is $\bar{A}$ non-vanishing where particle propagates? $\boxed{\text{HW 6.2}}$

$$\oint_{\substack{\text{circle}\\ \text{of radius } \rho}} \bar{A}\cdot\overline{dl} = \int_{\text{surface enclosed by loop}} (\bar{\nabla}\times\bar{A})\cdot d\bar{a}$$

$$= \int \underbrace{\bar{B}}_{\text{only for } \rho\le\rho_a}\cdot d\bar{a} = B\pi\rho_a^2 \neq 0$$

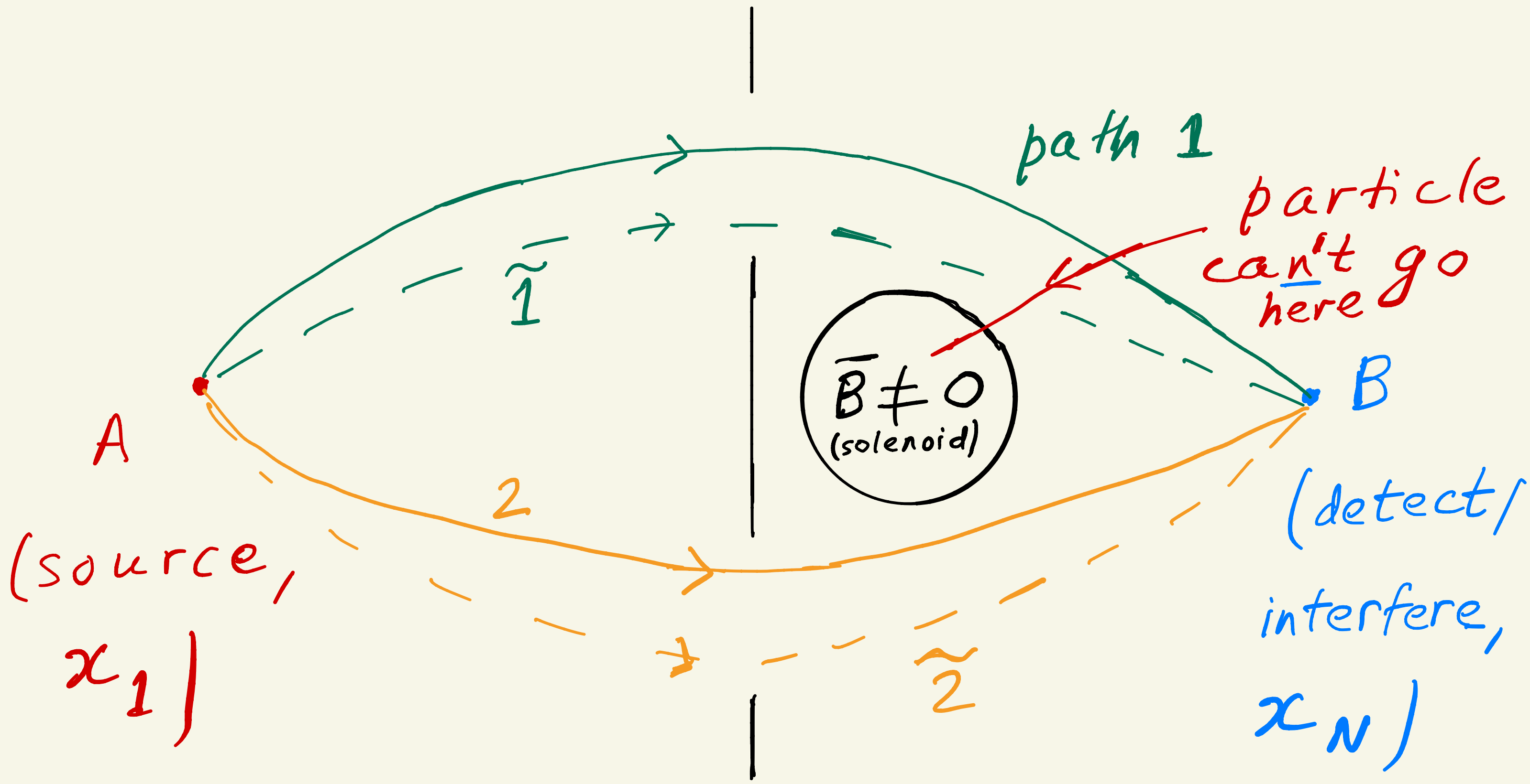
$\Rightarrow \bar{A} \neq 0$ inside shell (where particle propagates) ... circumferential

$\Rightarrow$ SE, energy eigenvalues modified

$$\bar{\nabla}' \longrightarrow \left(\bar{\nabla}' - \frac{ie}{\hbar c}\bar{A}\right)$$

... again, even though particle never feels Lorentz force (classically) [like electrostatic & gravitational examples from previous lecture]

Original AB effect



– Use PI : $L_{\text{classical}} = \frac{1}{2} m \dot{x}^2$ (no $\bar{B}$)

$$\longrightarrow L_{\text{classical}} + e/c\, \bar{A} \cdot d\bar{x}/dt$$

(Goldstein; Eq 1.63 sec. 1.5)

[equivalent to $H = \left(p - \frac{e}{c} A\right)^2 / 2m$]

– change in action for path segment:

$$S(n, n-1) \longrightarrow \underbrace{S(n, n-1)}_{\text{without } \bar{B}} + \underbrace{\frac{e}{c} \int_{t_{n-1}}^{t_n} dt \left(\frac{d\bar{x}}{dt}\right) \cdot \bar{A}}_{= e/c \int_{x_{n-1}}^{x_n} \bar{A} \cdot d\bar{s}}$$

$\Rightarrow$ contribution from one path:

$$\prod \exp\left[\frac{i\,S(n,n-1)}{\hbar}\right] \longrightarrow \left\{\prod \exp\left[\frac{i\,S(n,n-1)}{\hbar}\right]\right.$$

modulation due to $\bar{A}$

$$\left. \times \boxed{\exp\left(\frac{ie}{\hbar c}\int_{x_1}^{x_N} \bar{A}\cdot d\bar{s}\right)}\right\}$$

- $\boxed{\text{Sum}}$ over paths: consider two "above" $\bar{B}$ (1 and $\tilde{1}$), with

$$\int_{x_1}^{x_N} \bar{A}\cdot d\bar{s}\ (\text{along } 1) \underset{+}{-} \int_{x_1 \to x_N}^{x_N \to x_1} \bar{A}\cdot d\bar{s}\ (\text{along } \tilde{1} \to \text{"reverse"})$$

$$= \oint_{\text{closed loop}} \bar{A}\cdot d\bar{s} = \int_{\text{surface enclosed by loop}} \bar{\nabla}\times\bar{A}\cdot d\bar{a}$$

formed by paths 1 and reverse $\tilde{1}$

$$= \int \bar{B}\cdot d\bar{a} = 0$$

$\hookrightarrow$ vanishes...

$\Rightarrow$ $\boxed{\text{every}}$ path above $\bar{B}$ gives same modulation factor... similarly below $\bar{B}$

So, $\boxed{\text{new}}$ transition amplitude

$$= \left\{ \exp\left[\left(\frac{ie}{\hbar c} \right) \int_{x_1}^{x_N} \bar{A} \cdot d\bar{s} \right] \right\} \int \mathcal{D}[x(t)] \exp\left[\frac{i S(N,1)}{\hbar} \right]$$

to be determined & gauge-dependent

above

with no $\bar{B}$

+ above $\longrightarrow$ below

– probability to find particle in region B = $|\text{amplitude}|^2$

$\Rightarrow$ $\boxed{\text{interference}}$ between above & below paths depends on cos/sin of phase (modulation factor) $\boxed{\text{difference}}$ =

$$\int_{x_1}^{x_N} \bar{A} \cdot d\bar{s} \,(\text{above}) \underset{+}{-} \int_{x_1 \to x_N}^{x_N \to x_1} \bar{A} \cdot d\bar{s} \,(\text{below})$$

(reverse of path)

$= \oint \bar{A} \cdot d\bar{s}$ closed loop formed path by path above

Stokes theorem and reverse of below

$= \int (\bar{\nabla} \times \bar{A}) \cdot d\bar{a} = \int \bar{B} \cdot d\bar{a} = \phi_B$

(flux thru' solenoid)

independent of gauge choice

$\Rightarrow$ as $\boxed{\bar{B}}$ changed, sinusoidal behavior of particle detection, with

"period" $= \dfrac{2\pi\hbar c}{|e|}$