Lecture $\boxed{17}$, Oct. 9 (Fri.)

Outline for today / next week (1^{st} half)

— $\boxed{\text{Aharanov - Bohm}}$ (AB) effect : quantum interference with EM field

— "warm-up" first :

  — electrostatic potential

  — gravitational potential

[common thread of above 3 discussions: no force (classically) on particle, yet non-trivial effects ...]

— (More) "Preparing" for AB effect :

  — how to include EM potentials in classical & quantum Hamiltonian

  — gauge transformation [non-trivial for magnetic (vector) potential] : classical & quantum

Potential constant of x, t [trivial]

- Classically: $V(x) \to V(x) + \boxed{V_0}$:

independent of x, t

- $\boxed{\text{force}} = -\bar{\nabla} V$ unchanged...

$\Rightarrow$ $\bar{x}(t)$ $\boxed{\text{same}}$ ...

$\boxed{\text{QM}}$: $|\alpha, t_0; t)$ for $V \longrightarrow |\widetilde{\alpha, t_0; t}\rangle$ for $V + V_0$

(2 kets same - $|\alpha)$ - at $t = t_0$)

$$|\widetilde{\alpha, t_0; t}) = \exp\left[-i\left(\frac{p^2}{2m} + V + V_0\right)\left(\frac{t - t_0}{\hbar}\right)\right]|\alpha)$$

number

$$= \exp\left[\frac{-i V_0 (t - t_0)}{\hbar}\right] |\alpha, t_0; t)$$

$\boxed{\text{stationary}}$ state: time dependence

$$\exp\left[\frac{-iE(t - t_0)}{\hbar}\right] \longrightarrow \exp\left[-\frac{i(E + V_0)(t - t_0)}{\hbar}\right]$$

$\boxed{E \to E + V_0}$

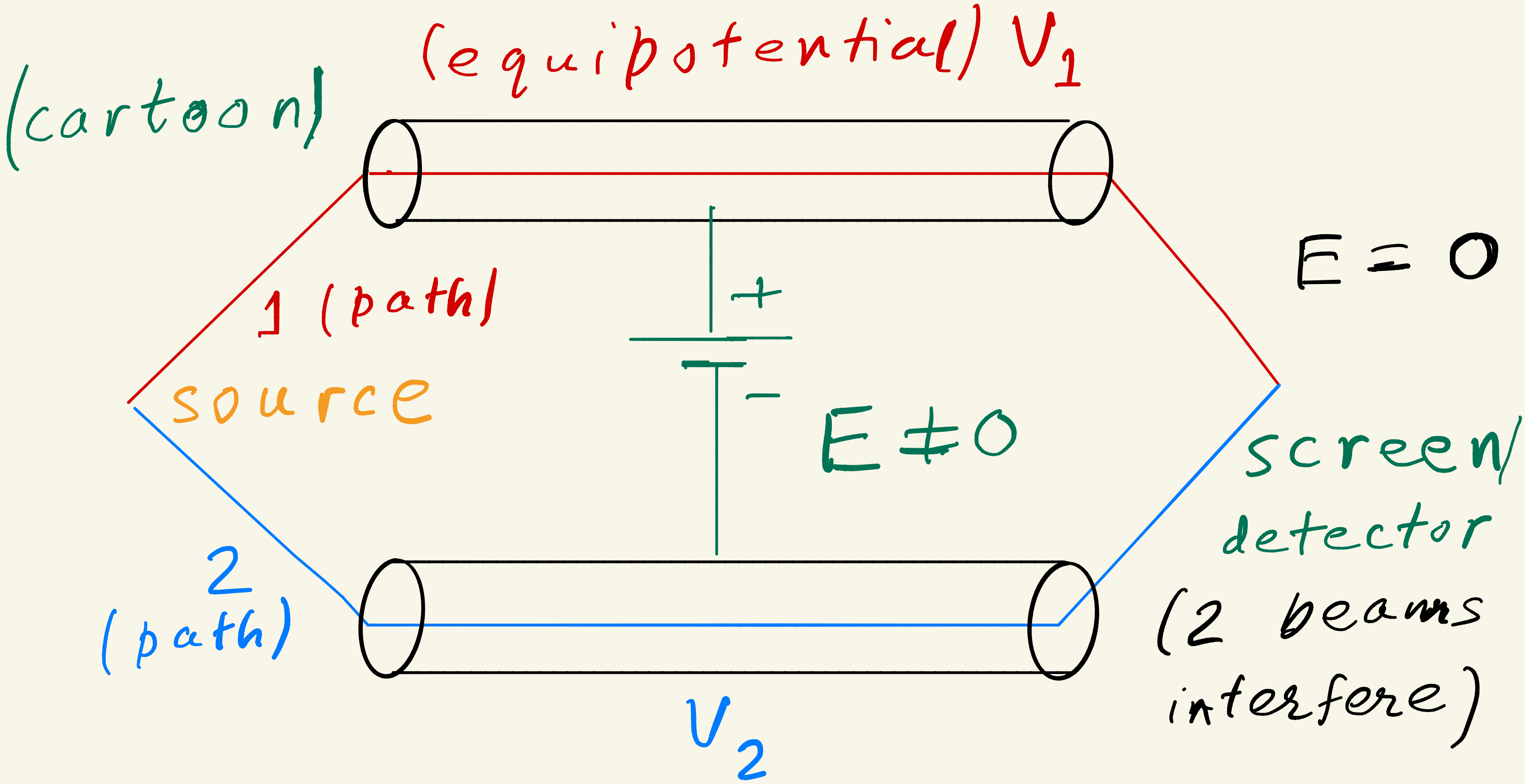
$\Rightarrow$ expectation values don't change (energy differences same)

$$\psi(x', t) \longrightarrow \exp\left[\frac{-iV_0(t - t_0)}{\hbar}\right] \psi(x', t)$$

[trivial e.g. of gauge transformation: different V giving same observable values]

Potential constant $\boxed{\text{along path}}$,
but depends on it ... ⇒ $\boxed{\text{interference}}$

$\boxed{\text{Electrostatic}}$ potential



$\boxed{\text{Wave-mechanically}}$ (roughly)

$$E = \frac{p^2}{2m} + V = \text{constant} \; (H \text{ is time-independent})$$

V ↳ different for 2 paths

⇒ p (slightly) different for 2 paths
(thus λ) ... ⇒ build-up phase difference
along 2 paths ... ⇒ interference

Or, $\boxed{\text{particle}}$ (wavepacket with spread $\ll$ tube dimension)

- no (classically) force inside tube
- each beam's phase shifts due to $V_{1,2}$

... $\Rightarrow$ $\boxed{\text{interference}}$ at screen $\propto$ cos/sin$(\phi_1 - \phi_2)$;

$$\boxed{(\phi_1 - \phi_2) = \frac{1}{\hbar}\int_{t_i}^{t_f} dt\,(V_2 - V_1)}$$

- $(V_2 - V_1)$ is observable QM effect: intensity on screen oscillates as $\boxed{(V_2 - V_1)}$ is dialed

($\hbar \to 0$: interference "washed out" due to oscillations)

$\boxed{\text{Gravitational}}$ potential

- Classically : $m\ddot{x} = -m\nabla\phi_{grav}$

$\Rightarrow$ acceleration independent of m

- QM : $\left[-\frac{\hbar^2}{2m^2}\nabla^2 + \frac{\cancel{m}}{\cancel{m}}\phi_{grav}\right] = \frac{i\hbar}{m}\frac{\partial\psi}{\partial t}$

$\Rightarrow \hbar/m$ remains (cf. classical)

- PI : $\langle x_n, t_n | x_{n-1}, t_{n-1}\rangle = \sqrt{\frac{m}{2\pi i\hbar\Delta t}} \times$

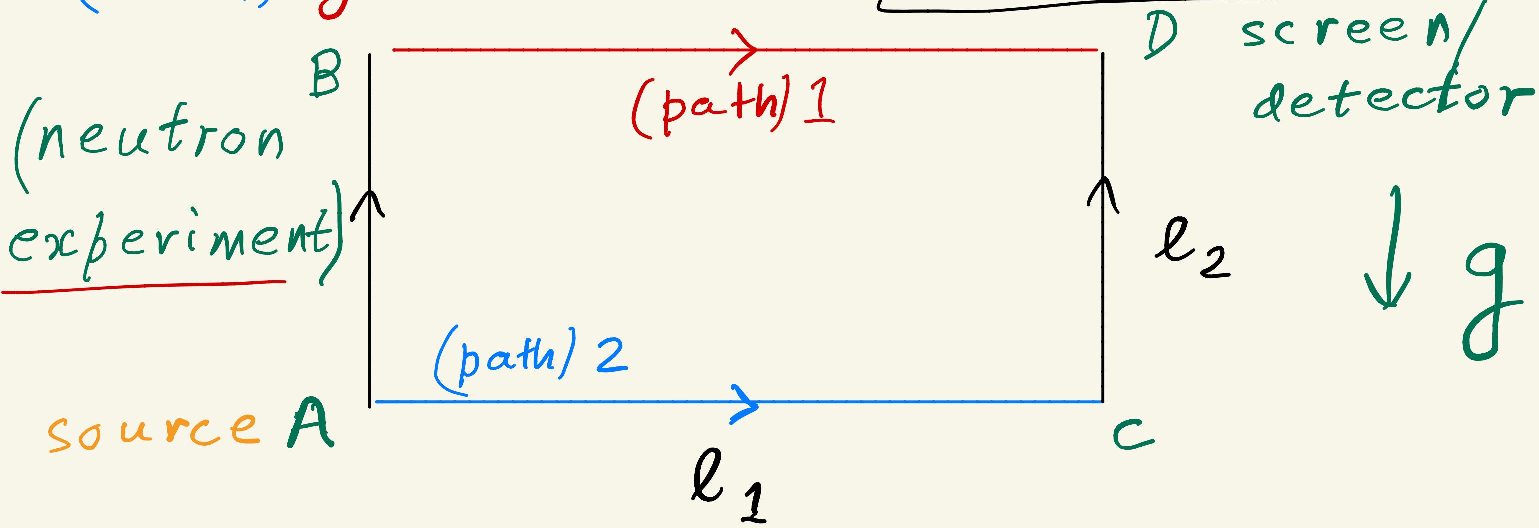
$$\exp\left[i\int_{t_{n-1}}^{t_n} dt\, \frac{\left(\frac{1}{2}m\dot{x}^2 - m\phi_{grav}\right)}{\hbar}\right]$$

- $\boxed{\text{Ehrenfest}}$ theorem (expectation values in QM obey classical laws):

$$d^2\langle x\rangle/dt^2 = -g\hat{z} \quad \left(\text{no } \hbar/m\right)$$

- QM ravitational effects not seen in energy-levels (since too weak)

– (QM) gravitational $\boxed{\text{interference}}$



– $\boxed{1^{st}}$ ABCD in $\boxed{\text{horizontal}}$ plane: ignore gravity

ABCD vertical

– Next: make ABCD vertical

[rotate about AC by $\delta = \pi/2$]

in general, $\delta \neq 0$

– gravity-induced phase (difference): so drop...

$\boxed{\text{AB vs. CD}}$ vertical: same phase shift (there is force)

– only $\boxed{\text{AC \& BD}}$ horizontal matter (no classical force)

phase difference

$$= \left[m\, g\, \ell_2 \sin\delta\, T / \hbar \right]$$

$$= \left(\frac{m^2}{\hbar^2} g \ell_1 \ell_2 \lambda\!\!\!^{-} \sin\delta \right)$$

$\lambda\!\!\!^{-}$

dial phase difference using tilt of ABCD

use $T = \ell / v$ of wave-packet

$\lambda\!\!\!^{-} = \hbar / mv$

- at quantum level, gravity not purely "geometric" (effect depends on $\hbar / m$: "washed out" for $\hbar \to 0$)

$\boxed{\text{Include}}$ EM field in $\boxed{\text{Hamiltonian}}$

$$\bar{E} = -\bar{\nabla}\phi \;:\; \bar{B} = \bar{\nabla}\times\bar{A}$$

as in warm-up

$$H = \frac{1}{2m}\Big(\underbrace{p}_{\text{canonical momentum conjugate to } x} - \frac{e}{c}A\Big)^2 + \overbrace{e\phi}^{x}$$

from $\boxed{\text{Goldstein Eq. 8.35}}$

$\boxed{\text{classical}}$

π (kinematical momentum) $= m\dfrac{d\bar{x}}{dt} =$
$= m\,\partial H/\partial p$ (Hamilton's equation) $= \left(p - \dfrac{e}{c}A\right)$

$\boxed{\text{QM}}$: ϕ, A are function of x (operator) $\Rightarrow$

$$[p, A] \neq 0 \quad \big([x, p] = i\hbar\big)$$

$$\left(p - eA/c\right)^2 \longrightarrow p^2 - \frac{e}{c}(p.A + A.p) + (e/c)^2 A^2$$

(so that H is Hermitian)

H-picture $\dfrac{dx}{dt} = \dfrac{1}{i\hbar}[x, H]$

$= \dfrac{(p - eA/c)}{m}$ $[x, p \cdot A]$ ("special" for EM)

from $[x, p^2]$ (usual)

$\Rightarrow$ $\boxed{p \text{ (operator)} \neq m\dfrac{d\bar{x}}{dt} = \text{mechanical momentum operator}}$

$[p_i, p_j] = 0$; $[\pi_i, \pi_j] \neq 0$

$= \left(\dfrac{i\hbar e}{c}\right) \varepsilon_{ijk} B_k$

$\Rightarrow$ $\boxed{H = \pi^2/(2m) + e\phi}$ ...

QM version of Lorentz force :

$\boxed{m\, d^2\bar{x}/dt^2} = d\bar{\pi}/dt =$ (next page)

$$e\left[\bar{E}+\frac{1}{2c}\left(\frac{d\bar{x}}{dt}\times\bar{B}-\bar{B}\times\frac{d\bar{x}}{dt}\right)\right]$$

order important, since $\left[\bar{B}(x), \frac{d\bar{x}}{dt}\right] \neq 0$

$\Rightarrow$ Ehrenfest's theorem

upon taking expectation value

$\boxed{\text{S - picture}}$ $\langle x'| i\hbar\frac{\partial}{\partial t}|\alpha\rangle = \langle x'|H|\alpha\rangle$

(start with SE for ket)

need to work out

$\langle x'|H|\alpha, t_0; t\rangle$

use $\langle x'|A(x) = A(x')\langle x'|$

$$= \langle x'|\left(p-\frac{e}{c}A\right)\left(p-\frac{eA}{c}\right)|\alpha, t_0; t\rangle$$

use $\langle x'|p|\beta\rangle = -i\hbar\frac{\partial}{\partial x'}\langle x'|\beta\rangle$

$$= \left[-i\hbar\frac{\partial}{\partial x'} - A(x')\right]\langle x'|\left(p-\frac{e}{c}A\right)|\alpha, t_0; t\rangle$$

["repeat" action of $(p - eA/c)$]

$$= \left[-i\hbar\frac{\partial}{\partial x'} - A\right]\left[-i\hbar\frac{\partial}{\partial x'} - A\right]\underbrace{\langle x'|\alpha, t_0; t\rangle}_{\psi_\alpha(x', t)}$$

acts here also

$$\boxed{\frac{1}{2m}\left[-i\hbar\nabla' - \frac{eA}{c}\right]\left[-i\hbar\nabla' - \frac{eA}{c}\right]\psi_\alpha(x', t) + e\phi\,\psi_\alpha(x', t) = i\hbar\frac{\partial}{\partial t}\psi_\alpha(x', t)}$$

continuity equation: $\dfrac{\partial\rho}{\partial t} + \nabla'\cdot j = 0$

$\rho = |\psi|^2$ (as before)

(new)

$$j \equiv \frac{\hbar}{m}\,\mathrm{Im}(\psi^* \nabla' \psi) - \left(\frac{e}{mc}\right) A|\psi|^2$$

(expected: $\nabla' \rightarrow \left[\nabla' - \left(\dfrac{ie}{\hbar c}\right)A\right]$)

$$= \rho/m \left(\bar{\nabla} S - \frac{eA}{c} \right) \qquad \left[\psi = \sqrt{\rho}\, \exp(iS/\hbar) \right]$$

$$\boxed{\int d^3x'\, \bar{j}} = \frac{\bar{p} - e\bar{A}/c}{m} = \frac{\bar{\pi}}{m}$$

(as desired : kinematical momentum)

- So, we have consistent way to incorporate EM potential in QM: reproduce Lorentz force (used H-picture); probability flux ($\bar{j}$) appropriately modified, using S-picture)...

Gauge transformation: redundancy in description of same physics

- $\phi \to \phi + \lambda$ (constant) ; $\bar{A} \to \bar{A}$ (trivial) done earlier

- $\boxed{\bar{A} \to \bar{A} + \bar{\nabla} \Lambda(x)}$ (non-trivial)

(crucial for magnetic monopole)