Lecture 16, Oct. 7 (Wed.)

Outline for today (& part of Fri.)

- finish PI approach to QM:
  - Feynman's ansatz for (infinitesimal) transition amplitude
  - justification: (1) recover (earlier) propagator for small (time) segment; (2) get classical path as $\hbar \to 0$; (3) satisfies Schroedinger's time-dependent wave equation

---

Feynman's "intution": for each (small) segment of path, assign

$$\boxed{\langle x_n, t_n | x_{n-1}, t_{n-1} \rangle \propto \exp[i S(n, n-1)/\hbar]}$$

where $\boxed{S_{n,n-1} \equiv \int_{t_{n-1}}^{t_n} dt \, L_{classical}(x, \dot{x})}$ (classical action)

More precisely (match dimensions):

$$\boxed{\underbrace{\langle x_n, t_n | x_{n-1}, t_{n-1} \rangle}_{\sim 1/\text{length}} = \underbrace{\frac{1}{w(\Delta t)}}_{\text{weight factor}} \exp[i S(n, n-1)/\hbar]}$$

$\boxed{1^{st}}$ (sanity) check : recover, for small $\boxed{\Delta t}$, free-particle propagator of earlier:

$$K_{\text{free}}(x'', t''; x', t') (= \langle x'', t'' | x', t' \rangle)$$

$$= \sqrt{\frac{m}{2\pi i\hbar (t''-t')}} \exp\left[\frac{im(x''-x')^2}{2\hbar(t''-t')}\right]$$

$$S(n, n-1) = \int_{t_{n-1}}^{t_n} \left\{\frac{1}{2} m \dot{x}^2 - V(x)\right\} dt$$

$$\approx \Delta t \left[\frac{1}{2} m \underbrace{\left(\frac{x_n - x_{n-1}}{\Delta t}\right)^2}_{\dot{x}^2} - V\left(\frac{x_n + x_{n-1}}{2}\right)\right]$$

$$\Rightarrow \langle x_n, t_n | x_{n-1}, t_{n-1} \rangle = \frac{1}{w(\Delta t)} \exp\left(\frac{im(x_n - x_{n-1})^2}{2\hbar \Delta t}\right)$$

a la Feynman

"reproduces" $\boxed{\exp \ldots}$ of earlier propagator

$$\Rightarrow \frac{1}{w(\Delta t)} = \sqrt{m/(2\pi i\hbar \Delta t)}$$

("matching" pre-factors) (but related)

[see Sakurai for alternate way to get $w(\Delta t)$, by only requiring Feynman's amplitude gives $\delta(x_n - x_{n-1})$ as $\Delta t \to 0$]

So, $\boxed{\text{Feynman's}}$ proposal :

$$\boxed{\langle x_n, t_n | x_{n-1}, t_{n-1} \rangle = \sqrt{\frac{m}{2\pi i\hbar \Delta t}} \exp\left[\frac{iS(n, n-1)}{\hbar}\right]}$$

$\boxed{2^{nd}}$ (sanity) check : for finite $(t_N - t_1)$, only classical path relevant as $\hbar \to 0$

– Using
$$\boxed{\langle x_n, t_n | x_{n-1} t_{n-1} \rangle = \sqrt{\frac{m}{2\pi i \hbar \Delta t}} \times \exp\left[i S(n, n-1)/\hbar\right]}$$
Feynman

for each segment of path (choice of $x_{N-1}, x_{N-2} \ldots x_2$), we get

$$\underbrace{\boxed{\langle x_N, t_N | x_1, t_1 \rangle}}_{\text{(total) transition amplitude ala Feynman}} = \lim_{\substack{N \to \infty \\ (\Delta t \to 0)}} \underbrace{\left(\frac{m}{2\pi i \hbar \Delta t}\right)^{\frac{N-1}{2}}}_{\text{weight factors}}$$
$$\times \underbrace{\int dx_{N-1} \int dx_{N-2} \cdots \int dx_2}_{\text{"sum" over paths}} \underbrace{\prod_{n=2}^{N}}_{\text{(product)}} \underbrace{\exp\left[\frac{i S(n, n-1)}{\hbar}\right]}_{\text{transition amplitude for segment of path}}$$

– Compact notation :
$$\int_{x_1}^{x_N} \mathcal{D}[x(t)] \equiv \lim_{N \to \infty} \left(\frac{m}{2\pi i \hbar \Delta t}\right)^{\frac{N-1}{2}} \left(\int dx_{N-1} \cdots \int dx_2\right)$$
$$\Delta t = (t_N - t_1)/(N-1)$$

– Also, $\prod_{n=2}^{N} \exp\left[\frac{i S(n,n-1)}{\hbar}\right] = \exp\left[\frac{i}{\hbar} \sum_{n=2}^{N} S(n,n-1)\right] = e^{\left[\frac{i}{\hbar} S(N,1)\right]}$

action for (full) path

$$\langle x_N, t_N | x_1, t_1 \rangle = \int_{x_1}^{x_N} \mathcal{D}[x(t)] \exp\left[i \overbrace{\int_{t_1}^{t_N} dt \frac{L_{\text{classical}}(x, \dot{x})}{\hbar}} \right]$$

$$\sim \sum_{\text{path}} \exp\left(i S_{\text{path}} / \hbar \right)$$

— For given path, contributions from neighboring paths (S different, even if slightly) as $\hbar \to 0$ $\boxed{\text{cancel}}$ each other (due to $\exp[iS/\hbar]$ oscillations)

... $\boxed{\text{except}}$ for path which is extremum of $\boxed{S}$: $\delta S = 0$ (classical path!) ... $\Rightarrow$ $\boxed{S_{\text{neighbor}} \simeq S_{\text{min.}} \text{ (classical)}}$

$\Rightarrow$ contributions $\boxed{\text{add}}$-up ...

$\Rightarrow$ as $\boxed{\hbar \to 0}$, sum over paths $\boxed{\text{dominated}}$ by classical path (as desired)

Alternative (to Schroedinger equation) formulation, based on (a) superposition principle (sum over paths); (b) composition property of transition amplitude and (c) reproduce classical path as $\hbar \to 0$

$\boxed{3^{rd}}$ check : for $V \not\equiv 0$, show that Feynman's formula for $\boxed{\langle x_N t_N | x_1 t_1 \rangle}$ satisfies Schroedinger time-dependent wave equation, i.e., same as propagator, $K(x_N, t_N; x_1; t_1)$ of before

— since we want $\partial/\partial t$ of $(x_N, t_N | x_1, t_1)$, "split" it into large t_1 to t_{N-1} and small t_{N-1} to t_N ("final") intervals:

$$\boxed{\langle x_N, t_N | x_1, t_1 \rangle = \int dx_{N-1} \langle x_N, t_N | x_{N-1}, t_{N-1} \rangle \times \langle x_{N-1}, t_{N-1} | x_1, t_1 \rangle}$$

— Use notation : $\xi = x_N - x_{N-1}$; $x_N \rightarrow x$; $t_{N-1} \rightarrow t$
$t_N = t + \Delta t$ (Δt small, but ξ can be large so far : see below)

so that

$$\boxed{\langle x, t + \Delta t | x_1, t_1 \rangle = \int d\xi \langle x, t + \Delta t | x - \xi, t \rangle \times \langle x - \xi, t | x_1, t_1 \rangle}$$

— For Δt small, we have $\boxed{1^{st}}$ factor on RHS

$$\langle x, t+\Delta t \,|\, x-\xi, t\rangle = \left.\sqrt{\frac{m}{2\pi i\hbar\Delta t}}\;\right\}\begin{array}{l}\text{weight} \\ \text{factor}\end{array} \times$$

a la Feynman

$$\exp\left[\underbrace{i\frac{m}{2\hbar}\underbrace{\frac{\xi^2}{\Delta t}}}_{\frac{i S_{N,N-1}}{\hbar}} - \frac{i}{\hbar}\int_t^{t+\Delta t} dt\, V\left(\begin{array}{c}\text{between } x \;\&\\ x-\xi\end{array}\right)\right]$$

$$\simeq \int \frac{(x_N - x_{N-1})^2}{(\Delta t)^2}\,dt$$

$$= \int dt\, \dot{x}^2$$

from 1st factor

— Use $\displaystyle\lim_{\Delta t\to 0}\sqrt{\frac{m}{2\pi i\hbar\Delta t}}\exp\left(\frac{im\xi^2}{2\hbar\Delta t}\right) = \boxed{\delta(\xi)}$

so that $\int d\xi \ldots$ dominated by ξ small in 1st factor

$\Rightarrow$ two expansions (2nd factor)

(a) $\langle x-\xi, t \,|\, x_1, t_1\rangle$ and $V\left(\begin{array}{c}\text{between } x \;\&\\ x-\xi\end{array}\right)$

in powers of ξ, but drop ξ^{odd}, since

$\times \exp\left[i\frac{m\xi^2}{2\hbar\Delta t}\right]$ (even power of ξ)...

$$\Rightarrow \int d\xi \ldots \to 0$$

(b). expand for small Δt on LHS and in $\exp\left[-\frac{i}{\hbar}\int_t^{t+\Delta t} dt\, V\right]$ in 1st RHS factor ... to give

$$\langle x,t|x_1,t_1\rangle + \Delta t \frac{\partial}{\partial t}\langle x,t|x_1,t_1\rangle + \Delta t^2 \ldots =$$

$$\sqrt{\frac{m}{2\pi i\hbar\Delta t}}\int d\xi \exp\left[\frac{im\xi^2}{2\hbar\Delta t}\right] \times$$ ← RHS 1st factor

$$\left[1 - \frac{i}{\hbar}\Delta t\, V(x) - \frac{i}{\hbar}\Delta t \frac{\xi^2}{2}\frac{\partial^2}{\partial x^2}V(x)\right]$$

1 ← call it term 1 ; 2 ; 3

drop ξ^1

$$\left[\langle x,t|x_1,t_1\rangle + \frac{\xi^2}{2}\frac{\partial^2}{\partial x^2}\langle x,t|x_1,t_1\rangle\right]$$ ← RHS 2nd factor

4 ; 5

– Use $\int d\xi \sqrt{\frac{m}{2\pi i\hbar\Delta t}}\exp\left[\frac{im\xi^2}{2\hbar\Delta t}\right] = 1$ to

(i). cancel leading terms [not $\propto \Delta t, \xi^2$] on both sides [4 × 1 on RHS] ... and

(ii). in evaluating 4 × 2

– Use $\sqrt{\frac{m}{2\pi i \hbar \Delta t}} \int d\xi \, \xi^2 \exp\left[\frac{im\xi^2}{2\hbar \Delta t}\right] = \frac{i\hbar}{m} \Delta t$

to evaluate 4×3 and 5×2

– We then get

$$\Delta t \, \partial/\partial t \langle x, t | x_1, t_1 \rangle + \Delta t^2 \ldots = -\frac{i}{\hbar} V(x) \Delta t \quad 4 \times 2$$

$$+ \left(i\hbar/m\right)\left(-\frac{i}{2\hbar}\right)(\Delta t)^2 \frac{\partial^2 V}{\partial x^2} \langle x, t | x_1, t_1 \rangle \quad 4 \times 3$$

$$+ \left(\frac{i\hbar}{2m}\right) \Delta t \, \partial^2/\partial x^2 \langle x, t | x_1, t_1 \rangle \left[1 - \frac{i}{\hbar} V(x) \Delta t\right]$$

1×5 $\qquad$ 5×2

– drop Δt^2 as $\Delta t \to 0$ (keep only Δt^1)

– what about 5×3 ?! It's even higher order in Δt (due to $\int d\xi \, \xi^4 \exp\ldots$)

$\ldots \times (i\hbar) \times 1/\Delta t$ gives (finally!)

$$i\hbar \frac{\partial}{\partial t}\langle x, t | x_1, t_1\rangle = V(x)\langle x, t | x_1, t_1\rangle - \frac{\hbar^2}{2m}\frac{\partial}{\partial x^2}\langle x, t | x_1, t_1\rangle$$

... as desired ...