Lecture $\boxed{15}$, $\boxed{\text{Oct. 5 (Mon.)}}$

Outline for today/this week

- continue PI approach to QM:
  - properties of $\boxed{\text{propagator}}$ (K):
    explicit form for free particle; "trace" of K; H-picture viewpoint (vs. S-picture so far)
  - onto "using" K in PI: basic idea; details (Feynman's proposal)

Explicit form for $\boxed{K(x', t'; x'', t_0)}$

$$\boxed{K =} \langle x' | \underbrace{U(t, t_0)}_{\exp\left[-iH(t-t_0)/\hbar\right]} | x'' \rangle$$

$$\exp\left[\frac{-iH(t-t_0)}{\hbar}\right] \underbrace{\sum_{a'} |a'\rangle\langle a'|}_{\mathbb{1}}$$ and $[A, H] = 0$

$|a'\rangle$ is energy eigenket $(E_{a'})$

gives $\exp\left[-iE_{a'}(t-t_0)/\hbar\right]$

$$\ldots = \boxed{\sum_{a'} \langle x' | a' \rangle \langle a' | x'' \rangle \exp\left[-iE_{a'}(t-t_0)/\hbar\right]}$$

Free particle: $H = p^2/(2m)$; choose $\boxed{A = p}$

$\langle x' | p' \rangle = 1/\sqrt{2\pi\hbar}\, \exp(ip'x'/\hbar)$ (plane-wave)

$\ldots \int d^3p'$ (for $\sum_{a'}$) $\ldots$ $\boxed{K = \sqrt{\dfrac{m}{2\pi i\hbar(t-t_0)}}\exp\left[\dfrac{im(x''-x')^2}{2\hbar(t-t_0)}\right] \text{ for free}}$

to be used in PI $\leftarrow$

(gives how Gaussian wavepacket spreads out...)

More "playing" with K: $G(t) \equiv \int d^3x' \underbrace{K(x', t; x', 0)}_{\langle x'|U(t,0)|x'\rangle}$ (set $x' = x''$; t_0 to 0)

is trace of U in x'-basis, but independent of basis $\Rightarrow$ use $|a'\rangle$ instead, where U is diagonal $\left[= e^{-iE_{a'}t/\hbar}\right]$ ...expect $\boxed{G = \sum_{a'} \exp(-iE_{a'}t/\hbar)}$

{ sanity check using explicit form of K above (in terms of $|a'\rangle$):

$G(t) = \sum_{a'} \underbrace{\int d^3x' \langle a'|x'' = x'\rangle\langle x'|a'\rangle}_{\mathbb{1}} \cdot \exp[-iE_{a'}(t-t_0)/\hbar] = \sum_{a'} \exp[-iE_{a'}(t-t_0)/\hbar]$

(with $\langle a'|a'\rangle = 1$) ...agrees... }

– Laplace-Fourier transform of G:

$$\tilde{G}(E) \equiv -i\int_0^\infty dt\, G(t)\exp(iEt/\hbar)/\hbar$$

"add": $\epsilon > 0$ ($\epsilon \to 0$ in end)

$$= -i\int dt \sum_{a'} \exp\left[-\frac{i(E_{a'} - E - i\epsilon)t}{\hbar}\right]/\hbar$$

$$= -i\sum_{a'} \frac{\exp\left[\dfrac{+i(E - E_{a'} + i\epsilon)t}{\hbar}\right]\Big|_0^\infty}{\dfrac{\hbar}{1}\left[\dfrac{i(E - E_{a'} + i\epsilon)}{\hbar}\right]}$$

$\infty \to$ gives 0 due to $\exp(-\epsilon t)$

"(math) trick" to suppress oscillations from $t \to \infty$

$$\Rightarrow \boxed{\tilde{G}(E) = +\sum_{a'} \left(\frac{1}{E - E_{a'}}\right) \quad (\epsilon \to 0)}$$

: energy eigenvalues are poles of $\tilde{G}(E)$ (Phys 624: QFT)

— Go to $\boxed{H}$-picture : so far, $\boxed{S}$-picture

$$K = \langle x' | U(t, t_0) | x' \rangle$$

Initial ket $|x'\rangle$ evolved to t (in S-picture)

bra dual to

($|x'\rangle$ is also H-picture eigenket at t_0)

$(U^\dagger |x'\rangle)$, which is base ket in H-picture at time $\boxed{t}$: $|x', t\rangle$

(Base kets vary with time in H-picture)

(so, $|x'\rangle = |x', t_0\rangle$)

$$\Rightarrow \boxed{K = \underbrace{\langle x', t|}_{\text{H-picture base bra at } t} \underbrace{x'', t_0\rangle}_{\text{H-picture base ket at } t_0}}$$

will use this notation henceforth

... reminds of unitary transformation in general, with matrix element :

$$\langle a^{(k)} | \boxed{U} | a^{(l)} \rangle = \underbrace{\langle a^{(k)}|}_{\text{old}} \underbrace{b^{(l)}\rangle}_{\text{new base ket}}$$

$\Rightarrow$ time evolution (due to K) is unitary transformation between 2 sets of base kets (of x operator, but different times)

Onto "using" K in PI

(Slight) "change" of notation:
$|x'', t_0\rangle \to |x'', t''\rangle$; $\langle x', t| \to \langle x', t'|$
and "switch": $\langle x', t' | x'', t''\rangle \to \langle x'', t'' | x', t'\rangle$

Basic idea: split finite time evolution into many "steps" (each choice is a path) ... sum over all allowed paths, e.g.,

$$\langle x''', t''' | x', t'\rangle = \int d^3x'' \langle x''', t''' | x'', t''\rangle \langle x'', t'' | x', t'\rangle$$

$(t''' > t'' > t')$

$\mathbb{1}$ (using H-picture base kets: time-dependent)

(composition property of transition amplitude)

[Also using S-picture instead:
$\langle x''' | U(t''', t') | x'\rangle = \langle x''' | U(t''', t'') U(t'', t') | x'\rangle$ (composition property of U)
fixed fixed

$$= \int d^3x' \langle x''' | U(t''', t'') | x''\rangle \langle x'' | U(t'', t') | x'\rangle = \int d^3x' \langle x''', t''' | x'', t''\rangle \times \langle x'', t'' | x', t'\rangle$$

$\mathbb{1}$

H picture kets/bras

$\Rightarrow$ Knowing $\langle x'', t'' | x', t'\rangle$ for infinitesimal time interval ($t'' = t' + dt$) gives – upon successive application – transition amplitude for finite time interval

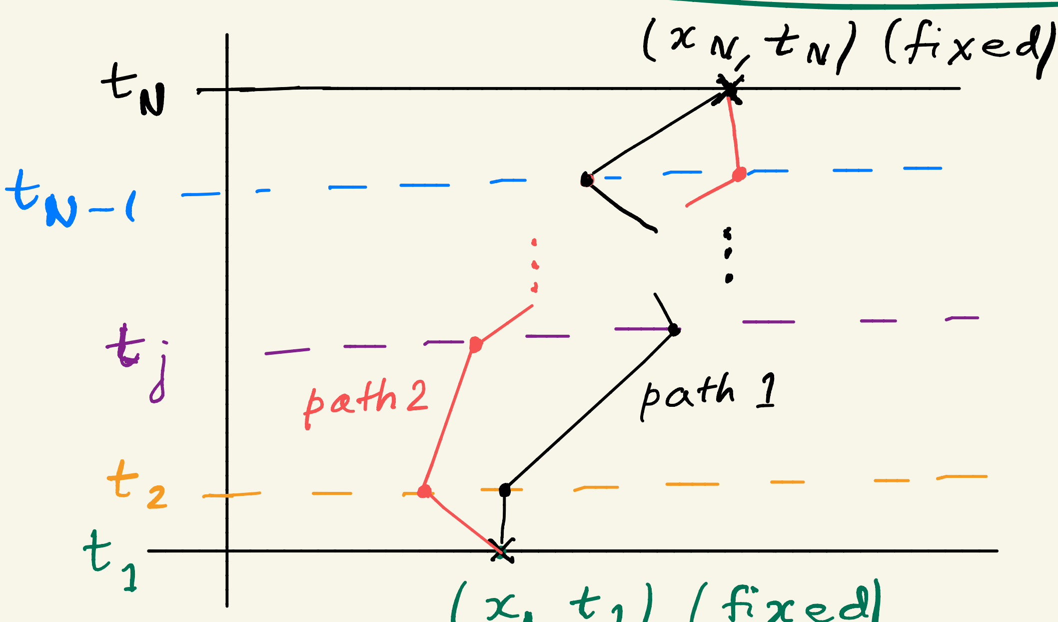
PI : full expression / picture

- use t_N, x_N for $t^{\text{"N primes"}}, x^{\text{"N primes"}}$
- divide t_1 to t_N into $(N-1)$ equal parts:

$$\Delta t = t_j - t_{j-1} = (t_N - t_1)/(N-1)$$

and use composition property

$$\Rightarrow \underbrace{\langle x_N, t_N | x_1, t_1 \rangle}_{\text{transition amplitude}} = \boxed{\int dx_{N-1} \int dx_{N-2} \cdots \int dx_2} \; \langle x_N, t_N | x_{N-1}, t_{N-1} \rangle \times \langle x_{N-1}, t_{N-1} | x_{N-2}, t_{N-2} \rangle \times \cdots \langle x_2, t_2 | x_1, t_1 \rangle$$



- Sum over paths (choice of intermediate $x_{2,\dots N-1}$):
for each path, multiply transition amplitudes for all (small) segments

- What about "$\int$ over" $\boxed{t_{1 \ldots N-1}}$? No because $\boxed{t}$ just a parameter (vs. x is operator)

- classically, $\boxed{L(x, \dot{x}) = \frac{1}{2} m \dot{x}^2 - V(x)}$

path from (x_1, t_1) to (x_N, t_N) is unique, minimizes action, $\boxed{S = \int_{t_1}^{t_N} dt \, L_{classical}(x, \dot{x})}$:

$\delta S = 0$

variation (small change in path, endpoints fixed)

..., vs. $\boxed{\text{all}}$ paths contribute to $\boxed{\text{QM}}$ transition amplitude

- How to get back classical physics from above expression for $\langle x_N, t_N | x_1, t_1 \rangle$ (as $\boxed{\hbar \to 0}$)?!

... enter Feynman ...