Lecture 14 Part II Oct. 2, 2020

more in HW 6.3, 6.4

Outline

- Examples of applying WKB approximation:
  SHO (again); bouncing ball (again)
- Another example of even/odd parity...
- Start Feynman's path integral approach to QM (propagator...): sec. 2.6 of Sakurai

x

WKB quantization condition:

$$\int_{x_1}^{x_2} dx \sqrt{2m\left[E - V(x)\right]} = \left(n + \tfrac{1}{2}\right)\pi\hbar$$

SHO: $V = \frac{1}{2} m\omega^2 x^2$

- turning points: $\pm\sqrt{2E/(m\omega^2)}$

$$2\int_0^{\sqrt{2E/(m\omega^2)}} dx\, \sqrt{2mE}\sqrt{1 - \frac{m\omega^2 x^2}{2E}} = \left(n + \tfrac{1}{2}\right)\pi\hbar$$

use $\sin\theta = \frac{\sqrt{m}}{\sqrt{2E}}\,\omega x$

$$= \frac{\pi E}{\omega} \Rightarrow \boxed{E = \left(n + \tfrac{1}{2}\right)\hbar\omega}$$

as before

Bouncing ball

— Exact solution : just choose odd $u_E(x)$ from linear potential $\Rightarrow$

$$\boxed{\varepsilon = \frac{E_n}{(\hbar^2 m^2 g^2)^{1/3}} = -\frac{1}{2^{1/3}} \underbrace{(\text{zeroes of } Ai)}_{<0}}$$

$$Ai\,(z < 0,\ |z| \text{ large}) \propto \cos\Big[\underbrace{\tfrac{2}{3}(-z)^{3/2} - \pi/4}_{(n+\frac{1}{2})\pi}\Big]$$

n = integer

$\Rightarrow$ (large (magnitude)) zeroes of Ai

$$\simeq (-)\left[\frac{\overbrace{(n+\tfrac{1}{2})\pi + \pi/4}^{(n+3/4)\pi}}{2/3}\right]^{2/3} : (n + 3/4) > 0 \Rightarrow$$

$n = 0, 1 \ldots$

... but "good" approximation only for $n \gg 1$

... so that $$\boxed{E_n \simeq \frac{(\hbar^2 m g^2)^{1/3}}{2}\left[3\left(n+\tfrac{3}{4}\right)\pi\right]^{2/3}} \quad (n = 0, 1 \cdots)$$

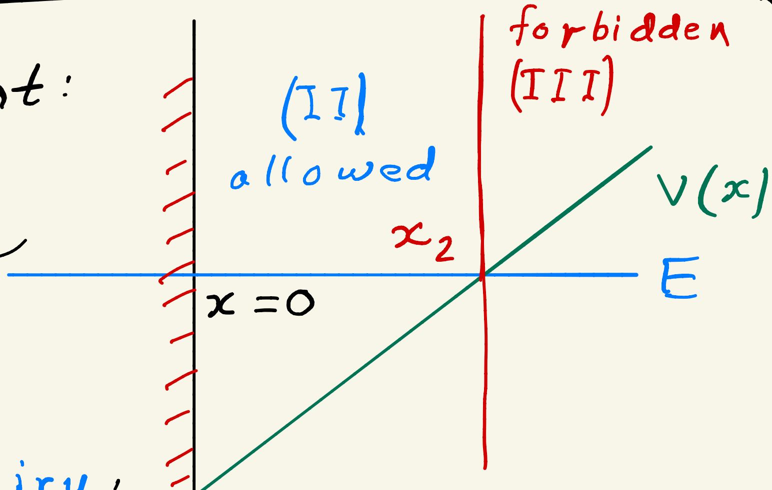
Use WKB approximation

$$: V = \begin{cases} mgx & \text{for } x > 0 \\ \infty & \text{for } x < 0 \end{cases}$$

— (single) turning point:

$x_2 = E/mg$

— use WKB solution for $x < x_2$ obtained by matching to Airy:



$$\propto \cos\left[+ \int_{x=0}^{x_2} dx' \sqrt{\frac{2m}{\hbar^2}(E - mgx)} - \pi/4 \right] = 0$$

$$\left[\frac{1}{\hbar}\,\frac{\sqrt{2m}}{mg}\,\frac{(-1)}{3/2}\,(E-mgx)^{3/2}\Big|_0^{E/mg} - \pi/4\right] = (n+1/2)\pi$$

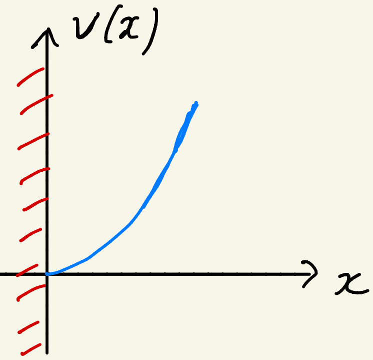
$$\frac{2}{3}\,\frac{\sqrt{2m}}{\hbar}\,\frac{E^{3/2}}{mg} = (n+3/4)\pi$$

$$\boxed{E = \left(\frac{mg^2\hbar^2}{2}\right)^{1/3}\left[3(n+3/4)\pi\right]^{2/3}}$$ ... as before

x

Another example of "re-using" solutions

$$V(x) = \begin{cases} \frac{1}{2}kx^2 & \text{for } x>0 \\ \infty & \text{for } x<0 \end{cases}$$



— $\boxed{u_E(x) = 0 \text{ at } x=0}$ ⇒ just choose odd solution of SHO

— E of ground state $= (\underset{n=1}{1} + \tfrac{1}{2})\hbar$

and $u_1(x)$ or $\langle x' | 1 \rangle$ } of SHO

$\boxed{\text{Path Integral}}$ approach by Feynman for QM (propagator ...)

- Motivation : useful to have another viewpoint (e.g., for QFT, especially gauge theories, in later courses; analyzing Aharonov-Bohm effect here), even if no new "results"

- central player : propagator : $K(x', t; x'', t_0)$

- $\boxed{\text{Basic}}$ idea : $\underbrace{|\alpha, t_0; t\rangle}_{\text{final}} = \boxed{U(t, t_0)} \underbrace{|\alpha, t_0\rangle}_{\text{initial}}$

not ket/ state

$$\boxed{U(t, t_0) = \exp\left[-iH(t-t_0)/\hbar\right]}$$

... try "similar" for $\boxed{\psi}$:

$$\psi(x', t) \sim \text{"operator" on } \psi(x'', t_0)$$

$$-\psi(x', t) = \overset{\text{basis}}{\langle x' |} \underbrace{\alpha, t_0; t\rangle}_{U(t,t_0)|\alpha, t_0\rangle} = \langle x' | U(t, t_0) \boxed{\int d^3x'' |x''\rangle} \underbrace{\langle x'' | \alpha, t_0\rangle}_{\psi(x'', t_0)}$$

$$\boxed{\psi(x', t) = \int d^3x'' \underbrace{\boxed{\langle x' | U(t, t_0) | x''\rangle}}_{\equiv K(x', t; x'', t_0)} \psi(x'', t_0)}$$

(Kernel of $\int$ operator)

- K depends on H (V), not on $\psi(x', t_0)$

— K is "all" you need to know to go from $\psi(x'', t_0)$ to $\psi(x', t)$

— $\boxed{K \text{ is matrix element of } U \text{ in } |x'\rangle \text{ basis}}$:

(as expected) ψ is $|\alpha\rangle$ in $|x'\rangle$ basis & $|\alpha\rangle$ is evolved by U ... $\Rightarrow$ ψ evolution due to U matrix element ...

— transition amplitude: $\langle b' | \underbrace{U(t, t_0) | a' \rangle}_{\text{time evolved } |a'\rangle}$ initial state ψ

(in general) for state probability to be in $|b'\rangle$ at t (A, B could be different operators)

$\Rightarrow$ $\boxed{K \text{ is amplitude for particle to "go" from } (x'', t_0) \text{ to } (x', t)}$

— Onto wavefunction (in x') interpretation of K

(1). $\boxed{\lim_{t \to t_0} K} = \langle x' | x'' \rangle$ $[U(t_0, t_0) = \mathbb{1}]$

$= \boxed{\delta^3(x' - x'')}$

wavefunction of particle exactly at x'' at t_0

(2) K (for $t > t_0$) $= \underbrace{\langle x' |}_{\text{basis}} \boxed{U(t, t_0) | x'' \rangle}$

ket localized at x'' at t_0, but now evolved it to t

$= \boxed{\text{wavefunction in } x' \text{ (in general, delocalized) of particle at } t \text{ which was initially (localized) at } x''}$

$\Rightarrow$ above expression for ψ follows : $\psi(x'', t_0)$ (in terms of K) $\neq \delta^3(x' - x'')$

So, "split" ψ at t_0 into various x'';

apply $\boxed{K}$ to each "δ-function" piece; then (sum) $\int$ over x''

... to get $\psi(x', t)$

$\Rightarrow$ $K(x', t; x'', t_0)$ – based on wavefunction picture – satisfies $\boxed{S's}$ time-dependent wave equation in x', t

for $\boxed{t > t_0}$: $\left[\frac{\hbar^2}{2m} \nabla'^2 + V(x') - i\hbar \frac{\partial}{\partial t} \right] K = 0$ for $t > t_0$

$\boxed{t < t_0}$: would like $K = 0$ for $t < t_0$ (no propagation "backward" in time)

$$\boxed{\left[-\frac{\hbar^2}{2m} \nabla'^2 + V(x') - i\hbar \frac{\partial}{\partial t} \right] K = -i\hbar\, \delta^3(x' - x'') \times \delta(t - t_0) \quad \text{for all } t}$$

i.e., K is Green's function for Schroedinger's time-dependent wave equation : particular solution (with δ-function "source") from which we can get general solution

check : equation valid for all $\boxed{t}$ reduces to earlier one for $t > t_0$ [since $\delta(t - t_0)$ on RHS $= 0$]

$$\int_{t_0-\epsilon}^{t_0+\epsilon\ (>0)} dt \ldots \quad \text{all terms} : \quad \text{RHS} \underbrace{= -i\hbar}_{\int dt}\, \delta^3(x'-x'')$$

$$\text{LHS} = \underbrace{\left[-\frac{\hbar^2}{2m}\nabla'^2 + V(x')\right] \underbrace{\overbrace{2\epsilon}^{\int dt}}_{\to 0 \text{ as } \epsilon \to 0} K}_{\to 0} \quad \left(\text{assuming } K \text{ continuous at } t_0\right)$$

$$- i\hbar \Big[\underbrace{K(x', t_0+\epsilon; x'', t_0)}_{\delta^3(x'-x'') \text{ as } \epsilon\to 0} - K(x', t_0-\epsilon; x'', t_0)\Big]$$

[matches RHS] $\ldots \Rightarrow K(x', t_0-\epsilon; x'', t_0) = 0$, thus also for $t < t_0$ (as desired)