Sep. 30 (Wed.) Lecture [13] & Oct. 2 (Fri.) Lecture [14] (Part I)

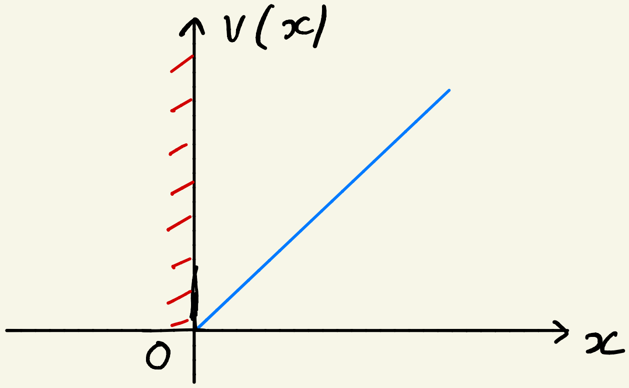
Outline for today / Fri.

— finish linear potential : "bouncing" ball (cold neutron experiment)

— onto WKB approximation

$\boxed{\text{Bouncing ball}}$ $k = mg$; x = height above floor ($x = 0$)

gravitational acceleration



$\Rightarrow$ $\boxed{x \leq 0}$ forbidden even QM!

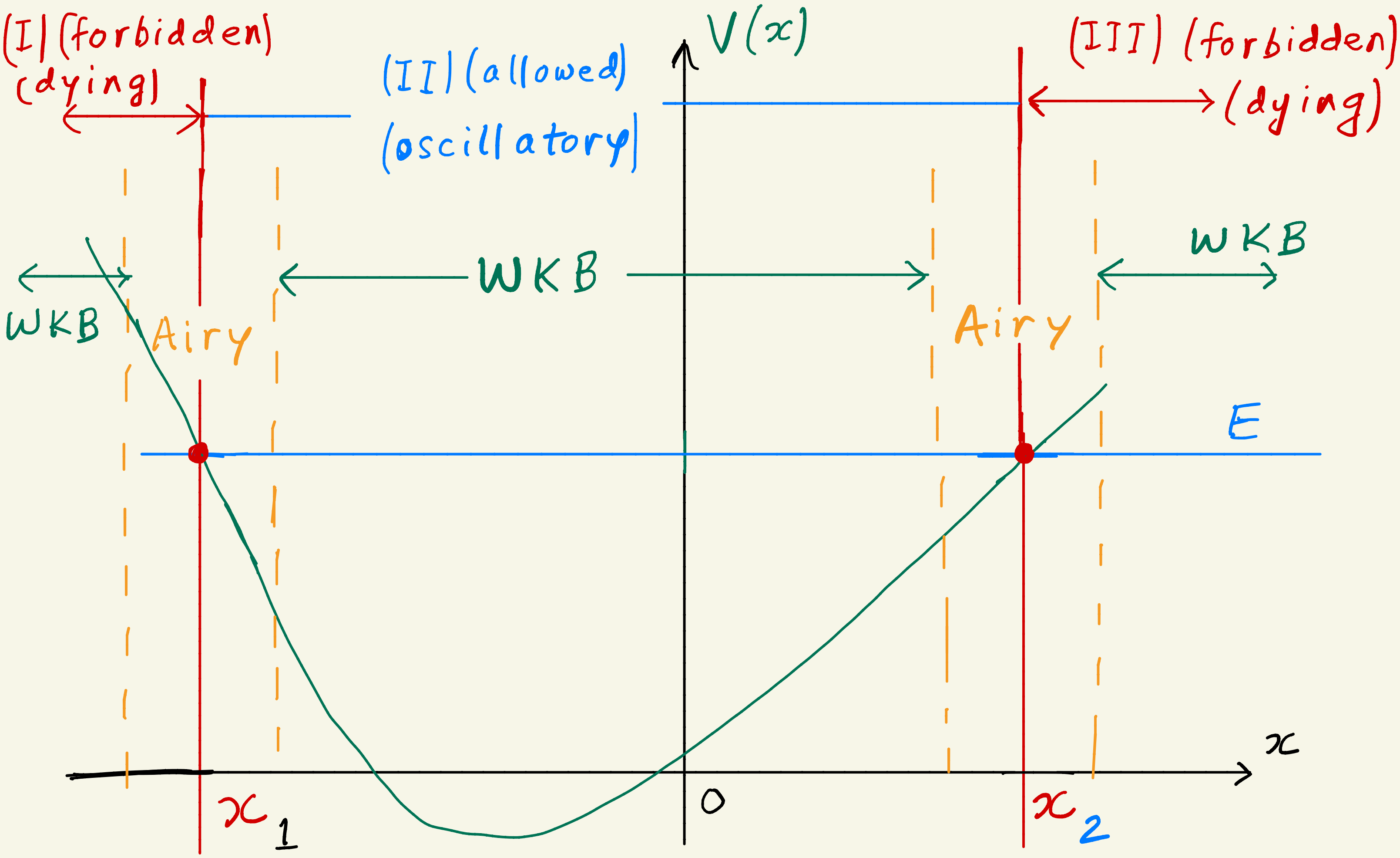
$\Rightarrow$ $u_E(x=0) = 0$ ... $\Rightarrow$ use $\boxed{\text{odd}}$ solutions from linear potential

so that $\boxed{E = -\frac{1}{2^{1/3}}\left[\text{zeroes of } Ai : -2.3, -4.1, -5.5 \ldots\right]}$

... $\boxed{\text{experiment}}$: cold neutrons (see Sakurai and original paper)

WKB approximation Sakurai: p. 110–116

Set-up: bound state from "potential-well" with (classical) turning points $x_{1,2}$ $(E = V)$:



Motivation/goal: find spectrum (E eigenvalues): exact solution difficult (unless SHO), so approximate

Basic/big picture idea "patchwork" of approximate solutions near turning points (Airy function) and away from them (WKB) ... ⇒ 2 WKB in middle... need to agree ... E quantized!

(More) details of $\boxed{\text{approach}}$:

$\boxed{\text{part}}$ (A): away from turning points ($x_{1,2}$)
["deep" in any region], de Broglie wavelength,
$\hbar/p \ll$ typical distance over which potential varies $\sim V / dV/dx$,

i.e., V "$\sim$" constant, giving $\boxed{\text{WKB}}$ solution:

region (II) [classically allowed: $E - V (\sim KE) > 0$]:
imaginary exponential (sin/cos: oscillatory)

vs. region (III) [classically forbidden: $E - V < 0$]:
real exponential (decaying or blowing up)
drop (not normalizable)

$\boxed{\text{Onto}}$ part (B): WKB breaks down
near turning points ($p \to 0 \Rightarrow \lambda$ not small) ...
... $V(x)$ approximated by linear
$\Rightarrow$ Airy function works, but
not valid away from turning points ...

... so, "stitch" together WKB (away from turning points) and Airy function (near turning points) ... will work! both oscillatory in allowed & decaying in fobidden

... so, what's "constraint"?!

— Near x_1 (but to right of it: allowed), choose specific combination of sin/cos of WKB to "match/dovetail with" Airy function ("fixed" cos form: see later or figure earlier)

— Similarly, near x_2 (but to left of it), "another" combination of sin/cos of WKB to give Airy ...

... two WKB solutions must have same form (both valid deep in region (II))

$\Rightarrow$ $\boxed{\text{WKB}}$ quantization condition:

$$\boxed{\int_{x_1}^{x_2} dx \sqrt{2m\left[E - V(x)\right]} = \left(n + \tfrac{1}{2}\right)\hbar\pi}$$

— X —

Onto really $\boxed{\text{detail}}$!

$\boxed{\text{Goal}}$: obtain E spectrum: Start with

part (A): developing $\boxed{\text{WKB}}$ solution

$$\frac{d^2 u_E}{dx^2} + \frac{2m}{\hbar^2}\left[E - V(x)\right] u_E(x) = 0$$

- with $\boxed{k(x)} \equiv \begin{cases} \left\{ 2m/\hbar^2 \left[E - V(x)\right] \right\}^{1/2} & \text{for } E > V(x) \text{ [allowed region (II)]} \quad \ldots (1)(a) \\ (-i)\left\{ \frac{2m}{\hbar^2}\left[V(x) - E\right] \right\}^{1/2} & \text{for } E < V(x) \text{ [forbidden region (I, III)]} \quad \ldots (1)(b) \end{cases}$

we get $$\frac{d^2 u_E}{dx^2} + \left[k(x)\right]^2 u_E(x) = 0 \quad \ldots (2)$$

- For $V(x) =$ constant, $u_E \sim e^{\pm ikx}$, so for "slowly varying" $V(x)$ (see below), use form $u_E(x) = \exp\left[i W(x)/\hbar\right] \ldots (3)$ so that

W → to be obtained

(2) gives $\left(\frac{dW}{dx}\right)^2 = \hbar^2 [k(x)]^2 + i\hbar \frac{d^2W}{dx^2}$ ... (4)

(So far, exact ... next, approximate / treat "perturbatively")

— Assume $\boxed{\hbar \left| \frac{d^2W}{dx^2} \right| \ll \left(\frac{dW}{dx}\right)^2}$ ... (5) :

requirement on $k(x)$ by plugging-in lowest-order solution $W_0(x)$ ... $\Rightarrow$ slowly varying potential

— Dropping 2^{nd} term on LHS of (4) gives

$$\boxed{W_0'(x) = \pm \hbar\, k(x)}$$

— "Consistency check" for above : plug $W_0'(x)$ into (5), with $W_0'' = \pm \hbar k'(x)$ to give

$$\left| \hbar^2 k'(x) \right| \ll \hbar^2 k^2(x) \quad \text{or} \quad \boxed{|k'| \ll k^2}$$

mathematical condition

— Now $|k(x)| = \left| \sqrt{\frac{2m}{\hbar^2}} \sqrt{E - V(x)} \right|$ gives

physical intuition

$$k'(x) = \ldots = \frac{m}{\hbar} \frac{dV/dx}{p}, \quad \text{where } \frac{p^2}{2m} \text{ (KE)} = E - V$$

— So, we require

$$\underbrace{\frac{m}{\hbar} \frac{dV/dx}{p}}_{k'} \ll \underbrace{\frac{2m}{\hbar^2} [E - V(x)]}_{k^2} \Rightarrow$$

$\frac{\hbar}{p} \sim \lambda$ (de Broglie wavelength) $\ll \dfrac{2(E-V)}{dV/dx}$

slowly varying potential on scale of wave-function $\sim \dfrac{V}{dV/dx}$ (assuming not close to turning point : $E \not\approx V$)

[~ distance over which $V(x)$ changes by $O(1)$]

Back to solving for $W(x)$: use $W_0(x)$ in 2nd term on RHS of (4) [vs. dropping it earlier... again ["like "doing perturbation theory"] to get $W_1(x)$ (next order) :

$$\left(\frac{dW_1}{dx}\right)^2 = \hbar^2 [k(x)]^2 + i\hbar W_0''(x)$$

$$\left[\text{or } \left(\frac{dW_0}{dx}\right)^2\right]$$

so that $W(x) \simeq W_1(x) = \pm \hbar \int^x dx' \left[k^2(x') \pm i\, k'(x')\right]^{1/2}$

$$\simeq \pm \hbar \int^x dx'\, k(x') \left[1 \pm \frac{i}{2} \frac{k'(x)}{k^2(x')}\right]$$

small due to (5) (see "consistency check")

after $\int$

$$= \pm \hbar \int^x dx'\, k(x') + \frac{i}{2} \ln [k(x)] \quad \cdots (6)$$

(from W_0)

plug into (3)

$$\Rightarrow \boxed{u_E \approx \exp\left[\frac{iW_1(x)}{\hbar}\right] = \exp\left[\pm i \int^x dx'\, k(x')\right] \frac{1}{[k(x)]^{1/2}}}$$

$\boxed{\text{Summary}}$ of part (A): "deep" in any region, u_E above (WKB solution) valid:

- region (I, III) (forbidden ($E < V(x)$; k is imaginary))

has $u_E \propto \exp\left[\pm \int^x dx' \sqrt{\frac{2m}{\hbar^2}\left[V(x) - E\right]}\right] \ldots$

as expected (choose "dying" ...)

→ region (II) [allowed ($E > V(x)$: k is real]:

two solutions (crucial) $\propto$

$$\boxed{\frac{1}{\left[E - V(x)\right]^{1/4}} \sin \text{ or } \cos\left(\int^x dx' \sqrt{\frac{2m}{\hbar^2}\left[E - V(x')\right]}\right) \ldots}$$

as expected (oscillatory)

Part (B): patchwork with Airy

$$Ai(z) \longrightarrow \begin{cases} \dfrac{1}{2\sqrt{\pi}} \dfrac{1}{z^{1/4}} \exp\left(-\dfrac{2}{3} z^{3/2}\right) & \text{for } z \to +\infty \\ & \text{dying (forbidden} \\ \dfrac{1}{\sqrt{\pi}} \dfrac{1}{(-z)^{1/4}} \cos\left[+\dfrac{2}{3}(-z)^{3/2} - \pi/4\right] & \text{for } z \to -\infty \\ & \text{oscillatory (allowed} \end{cases}$$

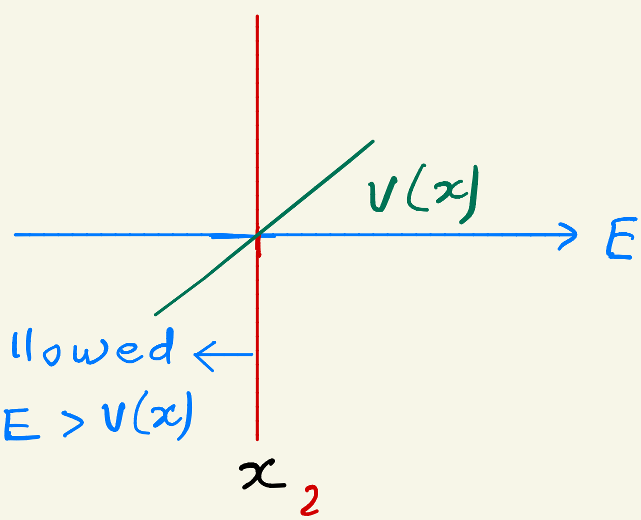
(see figure)

— Implement near $\boxed{x_2}$ $\left(\begin{array}{l} \text{to its left:} \\ x < x_2; \text{ allowed} \end{array} \right)$

$\boxed{\text{Claim}}$: choose following combination of sin & cos in WKB solution to match Airy:

$$\frac{1}{[E - V(x)]^{1/4}} \cos\left\{ + \frac{1}{\hbar} \int_x^{x_2} dx' \sqrt{\frac{2m\,[E - V(x')]}{\hbar^2}} - \frac{\pi}{4} \right\}$$

$\boxed{\text{Proof}}$ near x_2, we have $\boxed{E - V(x) \simeq \overbrace{V_0'}^{>0} \underbrace{(x_2 - x)}_{>0 \text{ for } x < x_2}}$



so that argument of sin or cos in WKB solution $\propto \displaystyle\int_x^{x_2} dx' \sqrt{[E - V(x')]}$

$$= \sqrt{V_0'} \int_x^{x_2} dx' \sqrt{x_2 - x} = \sqrt{V_0'} \frac{(-1)}{3/2} (x_2 - x)^{3/2} \Big|_x^{x_2}$$

$$= + \sqrt{V_0'}\, \tfrac{2}{3} (x_2 - x)^{3/2}$$

$\sim + \sqrt{V_0'}\, \tfrac{2}{3} (-z)^{3/2}$ in notation of Airy function

so that $z < 0$ ($x_2 > x$) is allowed

So, $\cos\left[\frac{2}{3}(-z)^{3/2} - \pi/4\right]$ of Airy

dovetails with $\cos\left[+\int_x^{x_2} dx'\sqrt{\frac{2m}{\hbar^2}[E-V(x')]} - \pi/4\right]$

of WKB solution (combination of sin & cos ...)

[Also, prefactor of sin or cos in WKB solution:

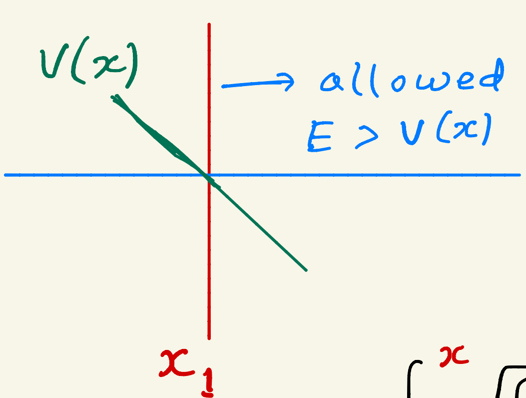
$$\frac{1}{[E-V(x)]^{1/4}} \sim \frac{1}{(x_2-x)^{1/4}} \sim \frac{1}{(-z)^{1/4}} \text{ ... of Airy}$$]

Similarly, near x_1 (but to right of it), we must choose WKB solution to be

$$\frac{1}{[E-V(x)]^{1/4}} \cos\left[+\int_{x_1}^{x} dx'\sqrt{\frac{2m}{\hbar^2}[E-V(x')]} - \frac{\pi}{4}\right]$$

in order to dovetail with Airy $(x > x_1)$

[For the sake of completeness(!), for x near x_1 (but $x > x_1$), we have $\boxed{E - V(x) = \overset{>0}{+V_0'}\underbrace{(x - x_1)}_{>0}}$



so that argument of sin or cos in WKB solution $\propto$

$$\int_{x_1}^{x}\sqrt{[E-V(x)]}\,dx' = \int_{x_1}^{x} dx'\sqrt{V_0'}(x-x_1)^{1/2}$$

$= \sqrt{V_0'}\ \frac{2}{3}(x - x_1)^{3/2}$ for $x > x_1$

$\sim \sqrt{V_0'}\ \frac{2}{3}(-z)^{3/2}$ of Airy function ($z < 0$ or $x > x_1$ is allowed)

So, $\cos\left[\frac{2}{3}(-z)^{3/2} - \pi/4\right]$ of Airy dovetails with $\cos\left[+\int_{x_1}^{x}\sqrt{\frac{2m}{\hbar^2}[E - V(x)]}\,dx' - \pi/4\right]$ combination of sin & cos of WKB solution ...

$\boxed{\text{Summary of part (B) so far}}$: in region (II), we have two WKB solutions, based on $x < x_2$:

$$\frac{1}{[E - V(x)]^{1/4}}\cos\Big[\underbrace{+\int_{x<x_2}^{x_2} dx'\sqrt{\frac{2m}{\hbar^2}[E - V(x)]}}_{\equiv\ \alpha(x)} - \pi/4\Big]$$

, while $x > x_1$ gives:

$$\frac{1}{[E - V(x)]^{1/4}}\cos\Big[\underbrace{+\int_{x_1}^{x\ > x_1} dx'\sqrt{\frac{2m}{\hbar^2}[E - V(x)]}}_{\equiv\ \beta(x)} - \pi/4\Big]$$

— quick check: both α, β (manifestly) > 0, as needed to match $(-z)^{3/2}$ of Airy (in allowed region)

— Above two cosines equal [both in region (II)] ... upto sign ...

$$\cos\left[\alpha - \pi/4\right] = \pm \cos\left[\beta - \pi/4\right]$$

even & odd

(1). $$(\alpha - \cancel{\pi/4}) = (\beta - \cancel{\pi/4}) + n\pi$$

(n = integer)

$$\underbrace{\frac{1}{\hbar}\int_{x}^{x_2}\dots + \int_{x}^{x_1} dx'\sqrt{2m\left[E - V(x')\right]}}_{\text{function of } x} = \underbrace{n\pi}_{\text{constant}}$$

... for fixed E, V, won't work [since RHS constant, but LHS function of x]

... "luckily", $\cos\theta = \cos(-\theta)$ ... $\Rightarrow$ another option:

(2). $$(\alpha - \pi/4) = -(\beta - \pi/4) + n\pi$$

note

$$\Rightarrow \quad \alpha + \beta = (n + 1/2)\pi$$

x-dependence "cancels"

$$\frac{1}{\hbar}\int_{\cancel{x}}^{x_2}\dots + \int_{x_1}^{\cancel{x}} dx'\sqrt{2m(E - V)} = (n + 1/2)\pi$$

$$\Rightarrow \boxed{\int dx'\sqrt{2m\left[E - V(x')\right]} = (n + 1/2)\pi\hbar}$$