Lecture 10 Sept. 23, Wed.

Outline for today (and Fri.)

- continue H-picture vs. S-picture
  - particle with potential : Ehrenfest theorem
  - base kets evolution
  - transition amplitudes
- "Energy-time" uncertainty relation
- Simple harmonic oscillator (SHO) via operator/algebraic method (not using Schroedinger's wave equation): energy eigenvalues, wavefunctions...

Particle with potential : $\boxed{H = \frac{p^2}{2m} + V(x)}$

(Use $[x, F(p)] = i\hbar\, \partial F/\partial p$ & $[p, G(x)] = -i\hbar \frac{\partial G}{\partial x}$)

$$dp/dt = \frac{1}{i\hbar}[p, V(x)] = -\partial V/\partial x$$

$$dx/dt = p/m \quad \text{(same as free)}$$

Combining, $d/dt\,(dx/dt) = d/dt\,(p/m) \cdots \Rightarrow$

$$\boxed{m\, d^2x/dt = -\partial/\partial x\, V} \quad \text{(operators)}$$

$$d/dt\,\langle p\rangle = d/dt\,\{\alpha|\, p^{(H)}\,|\alpha\rangle = \langle\alpha| -\partial V/\partial x\,|\alpha\rangle$$
$$= \langle -\partial V/\partial x\rangle$$

similarly $d/dt \langle x \rangle = \langle p \rangle / m \; \cdots \Rightarrow$ $\boxed{m \, d^2/dt \langle x \rangle = - \langle \partial V / \partial x \rangle}$

$\boxed{\text{Ehrenfest}}$ theorem : expectation values in QM obey classical laws

— Using $\boxed{S}$ picture instead :

$$d/dt \langle p \rangle = d/dt \Big(\underbrace{(\langle \alpha | U^{+})}_{\text{new}} \, p \, (U|\alpha\rangle) \Big)$$

fixed

use $U = \exp\left(-\frac{iHt}{\hbar}\right)$

$$= \langle \alpha | U^{+}\left(+\frac{iH}{\hbar}\right) p + p \left(-iH/\hbar\right) U |\alpha\rangle$$

$$= - \langle \alpha | U^{+} [p, H] U |\alpha\rangle \, i/\hbar$$

$$= - \left(\langle \alpha | U^{+} \right) \partial V / \partial x \left(U |\alpha\rangle \right) = - \langle \partial V / \partial x \rangle$$

Similarly, $d\langle x \rangle / dt = \langle p \rangle / m$ ... combining, get same as H-picture

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Base kets $\boxed{\text{evolution}}$ $\boxed{[A, H] \neq 0}$ $|a'\rangle$

$\boxed{S\text{-picture}}$: base kets (eigenkets of A) at time $\boxed{t}$ same as at $\boxed{t=0}$ (A unchanged)

... but if system starts out in $|a'\rangle$ (at $t=0$), then at later time t, it's in a different state ket : $\boxed{U|a'\rangle}$

$$|a'\rangle \longrightarrow U|a'\rangle \not\propto |a'\rangle$$

"time-evolved" base ket no longer base ket at later time t
S-picture

$\boxed{\text{H-picture}}$

$A|a'\rangle = a'|a'\rangle \Rightarrow$
(in S-picture)

$$U^{+} \left[A^{(S)} = A = A^{(H)}(0) \right] \underbrace{U U^{+}}_{1} |a'\rangle = a' \, U^{+} |a'\rangle$$

$A^{H}(t)$

$$A^{(H)}(t)\left(U^\dagger|a'\rangle\right) = \underbrace{a'}_{\text{eigenvalue}}\ \underbrace{\left(U^\dagger|a'\rangle\right)}_{\text{eigenket of } A^{(H)}(t)}$$

evolves in time

base kets at $t \neq$ those at $t=0$, i.e., $|a'\rangle$

$$\boxed{|a', t\rangle_H = U^\dagger|a'\rangle} \neq \text{time-evolved } |a'\rangle \text{ in } S\text{-picture} = U|a'\rangle$$

$$\boxed{i\hbar\, \partial/\partial t\, |a', t\rangle_H = -H|a', t\rangle_H}$$

"wrong" sign

If system starts out $|a'\rangle$ at $t=0$ [eigenket of $A^{(H)}(0)$], then remains in that state at later time t ... but then this is not base ket at time t in H-picture [which is $U^\dagger|a'\rangle$ as per above]

...as expected from general result for unitary equivalent operators: $A^{(S)}$ & $A^{(H)}$ have same eigenvalues (a'), but different eigenkets at time t: $\underbrace{|a'\rangle}_{\text{fixed}}$ vs. $\underbrace{U^\dagger|a'\rangle}_{\text{evolving}}$ in

- $\boxed{\text{transition amplitude}}$: system initially in $|a'\rangle$, what's probability to measure b' for B at later time t?

S-picture: $\underbrace{\left(\langle b'|\right)}_{\text{base bra (same as at } t=0)}\ \underbrace{\left(U|a'\rangle\right)}_{\text{time evolved } |a'\rangle}$

H-picture: $\underbrace{\left(\langle b'|U\right)}_{\text{base bra at time } t}\ \underbrace{\left(|a'\rangle\right)}_{\text{no time evolution}}$ ... same {sanity-check of base kets in H-picture}

not dagger

Summary of H- vs. S- pictures

	S-picture	H-picture
(general) state ket	evolving (in time)	fixed
operator (observable)	fixed	evolving
Base ket	fixed	evolving "oppositely" to state ket of S-picture

"Advantages" of H-picture (vs. S-picture):

"quicker" understanding of spreading of Gaussian wavepacket with time; Ehrenfest theorem in QM (expectation values obey classical physics laws); time-energy "uncertainty relation"...

$\boxed{\text{Time-energy}}$ uncertainty (alternate derivation in Sakurai)

$$\frac{d\langle Q\rangle}{dt} = \frac{i}{\hbar}\langle [H, Q]\rangle \; \ldots \Rightarrow$$

$$\langle(\Delta Q)^2\rangle \underbrace{\langle(\Delta H)^2\rangle}_{(\Delta E)^2} \geq \frac{1}{4}\left|\langle [Q, H]\rangle\right|^2 = \frac{\hbar^2}{4}\left|\frac{d\langle Q\rangle}{dt}\right|^2$$

eigenvalue of H

time is not operator, cf. x

define "Δt" by $\langle(\Delta Q)^2\rangle \equiv (\Delta t)^2 \left|\frac{d\langle Q\rangle}{dt}\right|^2$:

[Δt is time for $\langle Q\rangle$ to change by its dispersion ("fuzziness")]

... Combining, $\boxed{\Delta E \; \Delta t \geq \hbar/2}$: "similar" to

$\Delta x . \Delta p \geq \hbar/2$ but different interpretation (of

Δt vs. Δx) (usual) dispersion

as above