

## Homework 9

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## Goldstein 4.22

The colatitude<sup>1</sup>,  $\varphi$ , is defined in the following figure taken from Wikipedia ([http://en.wikipedia.org/wiki/File:Spherical\\_Coordinates\\_\(Colatitude,\\_Longitude\).svg](http://en.wikipedia.org/wiki/File:Spherical_Coordinates_(Colatitude,_Longitude).svg)).

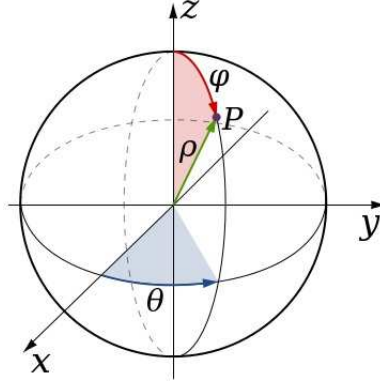


Figure 1: Definition of a point by colatitude,  $\varphi$ , longitude,  $\theta$ , and radius,  $\rho$ .

Suppose the projectile is launched horizontally<sup>2</sup>, along the  $\hat{e}_\varphi$  direction, at point P. The angular velocity of earth's rotation is  $\boldsymbol{\omega} = \omega \hat{z}$ , and the projectile's initial velocity is  $\mathbf{v} = v_0 \hat{e}_\varphi$ . The Coriolis acceleration is

$$\mathbf{a}_{cor} = -2(\boldsymbol{\omega} \times \mathbf{v}) \quad (1)$$

$$= -2\omega v_0 \hat{z} \times \hat{e}_\varphi \quad (2)$$

$$= -2\omega v_0 \sin(90^\circ + \varphi) \hat{e}_\theta \quad (3)$$

$$= -2\omega v_0 \cos \varphi \hat{e}_\theta \quad (4)$$

Since the acceleration is perpendicular to the initial velocity and points toward the right of the initial trajectory (in the northern hemisphere), the angular deviation is to the right in the northern hemisphere. In time  $\Delta t$ , the displacement due to the Coriolis acceleration (in the direction perpendicular to the initial velocity) is

$$\Delta s_{per} = \frac{1}{2}(2\omega v_0 \cos \varphi)(\Delta t)^2 \quad (5)$$

while the projectile covers a horizontal distance of

$$\Delta s_{hor} = v_0 \Delta t \quad (6)$$

<sup>1</sup>In the Goldstein problem,  $\theta$  and  $\varphi$  are interchanged.

<sup>2</sup>On the surface of a sphere, a "horizontal" direction is one which has no radial component, i.e. it is of the form  $c_1 \hat{e}_\theta + c_2 \hat{e}_\varphi$ .

So the angular deviation in time  $t$  is

$$\delta = \frac{\Delta s_{per}}{\Delta s_{hor}} = \omega \cos \varphi \Delta t \quad (7)$$

So the rate of angular deviation is

$$\omega \cos \varphi \quad (8)$$

which in Goldstein's notation for the colatitude, is  $\omega \cos \theta$ , as required. This result is first order in  $\omega$  because we have dropped the second order term  $\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$  in computing the deflection.

## Goldstein 4.24

Let  $(r, \theta)$  denote the polar coordinates of the bug at time  $t$ , where  $r$  is measured from the center of the wheel. The bug slips when the frictional force just equals the radial force acting on it. The radial force acting on the bug is

$$F_r = m(\ddot{r} - r\dot{\theta}^2) = -mr\dot{\theta}^2 \quad (\text{the bug crawls at constant speed}) \quad (9)$$

The tangential force on the bug is

$$F_\theta = m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 2mr\dot{\theta} \quad (\text{the wheel rotates at constant angular speed}) \quad (10)$$

The static friction force has a maximum value given by

$$f = \mu N \quad (11)$$

where  $\mu$  is the coefficient of static friction. If the bug crawls on the side of the spoke, the normal force which enters this expression is equal to the tangential force  $F_\theta$  for the bug not to fall off the spoke. In this case, slipping takes place when

$$\mu(2mr\dot{\theta}) = mr\dot{\theta}^2 \quad (12)$$

or equivalently

$$r = \frac{2\mu\dot{r}}{\dot{\theta}} = \frac{2 \times (0.30) \times (0.5 \text{ cm/s})}{3.0 \text{ rad/s}} = 0.1 \text{ cm} \quad (13)$$

But if the bug crawls on the top of the spoke, there are two normal reactions involved: the one that balances the gravitational force due to the bug's mass, and the other which equals the tangential force. In this case, the frictional force is modified to

$$f = \mu\sqrt{m^2g^2 + F_\theta^2} \quad (14)$$

so that the condition for slipping becomes

$$mr\dot{\theta}^2 = \mu m\sqrt{g^2 + 4\dot{r}^2\dot{\theta}^2} \quad (15)$$

which gives

$$r = \frac{\mu\sqrt{g^2 + 4\dot{r}^2\dot{\theta}^2}}{\dot{\theta}^2} = \frac{0.3\sqrt{(980 \text{ cm/s}^2)^2 + 4(0.5 \text{ cm/s})^2(3.0 \text{ rad/s})^2}}{(3.0 \text{ rad/s})^2} \approx 32.66 \text{ cm} \quad (16)$$

This result is intuitively obvious: if the bug crawls along the top of the spoke instead of the side, it can go much farther out before it starts to slip.

## Goldstein 4.25

The net force on the ball due to the rotation of the carousel is

$$\mathbf{F} = m(\ddot{r} - r\dot{\theta}^2)\hat{e}_r + m(r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta \quad (17)$$

As the ball is to remain stationary in the radial direction,  $\dot{r} = \ddot{r} = 0$ , so the force acting on the ball 6s after the carousel starts to move is

$$\mathbf{F} = -mr\dot{\theta}^2\hat{e}_r + mr\ddot{\theta}\hat{e}_\theta \quad (18)$$

$$\begin{aligned} &= -(3.0 \text{ kg})(7.0 \text{ m})(0.02 \times 2\pi \text{ rad/s}^2 \times 6\text{s})^2\hat{e}_r + (3.0 \text{ kg})(7.0 \text{ m})(0.02 \times 2\pi \text{ rad/s}^2)\hat{e}_\theta \\ &= -11.938 \text{ N}\hat{e}_r + 2.638 \text{ N}\hat{e}_\theta \end{aligned} \quad (19)$$

This is the force that the girl must provide to keep the ball moving in the circular path. Thus,

$$\mathbf{F}_{\text{girl}} = -11.938 \text{ N}\hat{e}_r + 2.638 \text{ N}\hat{e}_\theta \quad (20)$$

So, the girl must exert a force equal to 12.225 N directed at an angle  $\alpha$  given by

$$\alpha = \tan^{-1} \frac{2.638}{11.938} = \tan^{-1}(0.2209) = 12.46^\circ \quad (21)$$

The direction of the force that the girl must provide is shown in the figure below. It should be directed

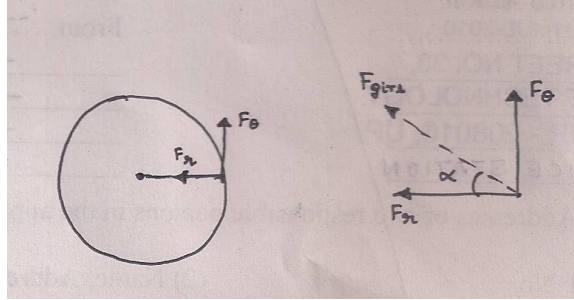


Figure 2: Figure for problem 4.25

at an angle of  $12.46^\circ$  measured clockwise from the line joining the ball to the center, or equivalently at an angle of  $167.54^\circ$  measured counterclockwise from the radius vector of the ball.

## Problem 1

### Part (a)

$$\frac{d\overleftrightarrow{R}_z}{d\theta} = \begin{pmatrix} -\sin\theta & \cos\theta & 0 \\ -\cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (22)$$

$$= \underbrace{\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\overleftrightarrow{R}_z(\theta)} \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{-M_z} \quad (23)$$

$$= -\overleftrightarrow{R}_z(\theta)M_z \quad (24)$$

$$= -M_z\overleftrightarrow{R}_z \quad (25)$$

The solution to this equation is

$$\boxed{\overleftrightarrow{R}_z(\theta) = e^{-\theta M_z}} \quad (26)$$

where the multiplicative constant is fixed by the requirement that  $\overleftrightarrow{R}_z(0) = I$ , the  $3 \times 3$  identity matrix.

### Part (b)

$$\text{tr}(\overleftrightarrow{R}_z(\theta)) = \text{tr} \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (27)$$

$$= 2\cos\theta + 1 \quad (28)$$

so

$$\cos\theta = \frac{1}{2}\text{tr}(\overleftrightarrow{R}_z(\theta)) - \frac{1}{2} \quad (29)$$

## Problem 2

### Part (a)

$$\overleftrightarrow{R}_{\hat{n}} = \overleftrightarrow{R}_z(\phi)\overleftrightarrow{R}_y(\theta) \quad (30)$$

$$= \underbrace{\begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\overleftrightarrow{R}_z(\phi)} \underbrace{\begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}}_{\overleftrightarrow{R}_y(\theta)} \quad (31)$$

$$= \begin{pmatrix} \cos\theta\cos\phi & \sin\phi & \sin\theta\cos\phi \\ -\cos\theta\sin\phi & \cos\phi & -\sin\theta\sin\phi \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \quad (32)$$

So,

$$\begin{aligned} \overleftrightarrow{R}_{\hat{n}} \overleftrightarrow{M}_z \overleftrightarrow{R}_{\hat{n}}^T &= \underbrace{\begin{pmatrix} \cos \theta \cos \phi & \sin \phi & \sin \theta \cos \phi \\ -\cos \theta \sin \phi & \cos \phi & -\sin \theta \sin \phi \\ -\sin \theta & 0 & \cos \theta \end{pmatrix}}_{\overleftrightarrow{R}_{\hat{n}}} \times \underbrace{\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{\overleftrightarrow{M}_z} \times \\ &\quad \underbrace{\begin{pmatrix} \cos \theta \cos \phi & -\cos \theta \sin \phi & -\sin \theta \\ \sin \phi & \cos \phi & 0 \\ \sin \theta \cos \phi & -\sin \theta \sin \phi & \cos \theta \end{pmatrix}}_{\overleftrightarrow{R}_{\hat{n}}^T} \end{aligned} \quad (33)$$

$$= \begin{pmatrix} 0 & -\cos \theta & -\sin \theta \sin \phi \\ \cos \theta & 0 & -\sin \theta \cos \phi \\ \sin \theta \sin \phi & \sin \theta \cos \phi & 0 \end{pmatrix} \quad (34)$$

Now,

$$\cos \theta \overleftrightarrow{M}_z = \begin{pmatrix} 0 & -\cos \theta & 0 \\ \cos \theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (35)$$

$$\sin \theta \cos \phi \overleftrightarrow{M}_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\sin \theta \cos \phi \\ 0 & \sin \theta \cos \phi & 0 \end{pmatrix} \quad (36)$$

$$\sin \theta \sin \phi \overleftrightarrow{M}_y = \begin{pmatrix} 0 & 0 & \sin \theta \sin \phi \\ 0 & 0 & 0 \\ -\sin \theta \sin \phi & 0 & 0 \end{pmatrix} \quad (37)$$

So, from Eqs. (34-37),

$$\overleftrightarrow{R}_{\hat{n}} \overleftrightarrow{M}_z \overleftrightarrow{R}_{\hat{n}}^T = \cos \theta \overleftrightarrow{M}_z + \sin \theta \cos \phi \overleftrightarrow{M}_x - \sin \theta \sin \phi \overleftrightarrow{M}_y \quad (38)$$

$$= \hat{n} \cdot \overleftrightarrow{M} \quad (39)$$

where  $\hat{n} = (\sin \theta \cos \phi, -\sin \theta \sin \phi, \cos \theta)^T$  and  $\overleftrightarrow{M} = (M_x, M_y, M_z)$ .

**Note:** The minus sign appearing in equation (38) is simply due to the sign convention adopted for  $\overleftrightarrow{R}_z(\phi)$  in equation (31). If we instead took

$$\overleftrightarrow{R}_z(\phi) = \begin{pmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (40)$$

which corresponds to a *clockwise* rotation about the  $z$ -axis through an angle  $\phi$  in the  $x-y$  plane, then we would get

$$\overleftrightarrow{R}_{\hat{n}} \overleftrightarrow{M}_z \overleftrightarrow{R}_{\hat{n}}^T = \cos \theta \overleftrightarrow{M}_z + \sin \theta \cos \phi \overleftrightarrow{M}_x + \sin \theta \sin \phi \overleftrightarrow{M}_y = \hat{n} \cdot \overleftrightarrow{M} \quad (41)$$

with  $\hat{n} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)^T$ . Therefore,

$$\boxed{\hat{n} \cdot \overleftrightarrow{M} = \overleftrightarrow{R}_{\hat{n}} \overleftrightarrow{M}_z \overleftrightarrow{R}_{\hat{n}}^T} \quad (42)$$

as required to be proved.

### Part (b)

As  $\overleftrightarrow{R}_{\hat{n}}$  is an orthogonal matrix,  $\overleftrightarrow{R}_{\hat{n}} \overleftrightarrow{R}_{\hat{n}}^T = 1$ . Therefore,

$$\overleftrightarrow{R}_{\hat{n}} \left( \exp(-\Phi \overleftrightarrow{M}_z) \right) \overleftrightarrow{R}_{\hat{n}}^T = \overleftrightarrow{R}_{\hat{n}} \left( \sum_{k=0}^{\infty} \frac{(-\Phi \overleftrightarrow{M}_z)^k}{k!} \right) \overleftrightarrow{R}_{\hat{n}}^T \quad (43)$$

$$= \sum_{k=0}^{\infty} \frac{(-\Phi)^k}{k!} \left[ \underbrace{(\overleftrightarrow{R}_{\hat{n}} \overleftrightarrow{M}_z \overleftrightarrow{R}_{\hat{n}}^T) \dots (\overleftrightarrow{R}_{\hat{n}} \overleftrightarrow{M}_z \overleftrightarrow{R}_{\hat{n}}^T)}_{k \text{ times}} \right] \quad (44)$$

$$= \sum_{k=0}^{\infty} \frac{(-\Phi)^k}{k!} \left[ \hat{n} \cdot \overleftrightarrow{M} \right]^k \quad (\text{using Eqn. (42)}) \quad (45)$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} \left[ -\Phi \hat{n} \cdot \overleftrightarrow{M} \right]^k \quad (46)$$

$$= \exp(-\Phi \hat{n} \cdot \overleftrightarrow{M}) \quad (47)$$

Therefore,

$$\boxed{\overleftrightarrow{R} = \overleftrightarrow{R}_{\hat{n}} \left( \exp(-\Phi \overleftrightarrow{M}_z) \right) \overleftrightarrow{R}_{\hat{n}}^T} \quad (48)$$

### Part (c)

From the result of problem 1,

$$\exp(-\Phi \overleftrightarrow{M}_z) = \begin{pmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (49)$$

and from equation (32) in part (a) of this problem,

$$\overleftrightarrow{R}_{\hat{n}} = \begin{pmatrix} \cos \theta \cos \phi & \sin \phi & \sin \theta \cos \phi \\ -\cos \theta \sin \phi & \cos \phi & -\sin \theta \sin \phi \\ -\sin \theta & 0 & \cos \theta \end{pmatrix} \quad (50)$$

Substituting into equation (48) we get

$$\begin{aligned} \overleftrightarrow{R} &= \underbrace{\begin{pmatrix} \cos \theta \cos \phi & \sin \phi & \sin \theta \cos \phi \\ -\cos \theta \sin \phi & \cos \phi & -\sin \theta \sin \phi \\ -\sin \theta & 0 & \cos \theta \end{pmatrix}}_{\overleftrightarrow{R}_{\hat{n}}} \times \underbrace{\begin{pmatrix} \cos \Phi & \sin \Phi & 0 \\ -\sin \Phi & \cos \Phi & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\exp(-\Phi \overleftrightarrow{M}_z)} \times \\ &\quad \underbrace{\begin{pmatrix} \cos \theta \cos \phi & -\cos \theta \sin \phi & -\sin \theta \\ \sin \phi & \cos \phi & 0 \\ \sin \theta \cos \phi & -\sin \theta \sin \phi & \cos \theta \end{pmatrix}}_{\overleftrightarrow{R}_{\hat{n}}^T} \end{aligned} \quad (51)$$

That is,

$$\overleftrightarrow{R} = \begin{pmatrix} \cos^2 \phi (\cos^2 \theta \cos \phi + \sin^2 \theta) + \cos \phi \sin^2 \phi & -\sin^2 \theta \sin[2\phi] \sin\left[\frac{\Phi}{2}\right]^2 + \cos \theta \sin \phi & \sin \theta (-\cos \theta \cos \phi (-1 + \cos \phi) + \sin \phi \sin \phi) \\ -\sin^2 \theta \sin[2\phi] \sin\left[\frac{\Phi}{2}\right]^2 - \cos \theta \sin \phi & \cos^2 \phi \cos \phi + (\cos^2 \theta \cos \phi + \sin^2 \theta) \sin^2 \phi & \sin \theta (\cos \theta (-1 + \cos \phi) \sin \phi + \cos \phi \sin \phi) \\ -\sin \theta (\cos \theta \cos \phi (-1 + \cos \phi) + \sin \phi \sin \phi) & \sin \theta (\cos \theta (-1 + \cos \phi) \sin \phi - \cos \phi \sin \phi) & \cos^2 \theta + \cos \phi \sin^2 \theta \end{pmatrix}$$

### Problem 3

#### Part (a)

Using the explicit form of  $\overleftrightarrow{R}$  derived above, via Mathematica<sup>3</sup>, we find that

$$\overleftrightarrow{R} \cdot \underbrace{\begin{pmatrix} \sin \theta \cos \phi \\ -\sin \theta \sin \phi \\ \cos \theta \cos \phi \end{pmatrix}}_{\hat{n}} = \begin{pmatrix} \sin \theta \cos \phi \\ -\sin \theta \sin \phi \\ \cos \theta \cos \phi \end{pmatrix} \quad (52)$$

So,  $\hat{n}$  is an eigenvector<sup>4</sup> of  $\overleftrightarrow{R}$  with eigenvalue +1.

#### Part (b)

Using Mathematica, we find that

$$\text{tr}(\overleftrightarrow{R}) = 1 + 2 \cos \Phi \quad (53)$$

so that

$$\cos \Phi = \frac{1}{2} \text{tr}(\overleftrightarrow{R}) - \frac{1}{2} \quad (54)$$

<sup>3</sup>Please refer to the file, `problem3.nb`, containing calculations for this part.

<sup>4</sup>A similar statement holds for the alternate form of  $\hat{n}$  mentioned on page 5, corresponding to the alternate sign convention for  $\phi$ .

## Problem 4

Using the results of the previous parts,  $\overleftrightarrow{R}$  corresponds to a rotation about an angle  $\Phi$  such that

$$\cos \Phi = \frac{1}{2} \text{tr}(\overleftrightarrow{R}) - \frac{1}{2} \quad (55)$$

and the axis of rotation is the eigenvector of  $\overleftrightarrow{R}$  which has an eigenvalue  $+1$ .

Using Mathematica<sup>5</sup>, we find

$$\overleftrightarrow{R} = \begin{pmatrix} -\frac{1}{8} & \frac{3\sqrt{3}}{8} & \frac{3}{4} \\ -\frac{3\sqrt{3}}{8} & -\frac{5}{8} & \frac{\sqrt{3}}{4} \\ \frac{3}{4} & -\frac{\sqrt{3}}{4} & \frac{1}{2} \end{pmatrix} \quad (56)$$

So,  $\text{tr}(R) = -1/4$ . The eigenvalues are

$$\lambda_1 = \frac{1}{8} (-5 + i\sqrt{39}) \quad (57)$$

$$\lambda_2 = \frac{1}{8} (-5 - i\sqrt{39}) \quad (58)$$

$$\lambda_3 = 1 \quad (59)$$

The matrix of eigenvectors is

$$\mathbf{V} = \begin{pmatrix} -\frac{3}{2} & -\frac{3}{2} & \frac{2}{3} \\ -\frac{i\sqrt{13}}{2} & \frac{i\sqrt{13}}{2} & 0 \\ 1 & 1 & 1 \end{pmatrix} \quad (60)$$

The normalized eigenvector corresponding to the eigenvalue  $+1$  corresponds to the axis of rotation,  $\hat{n}$ , and is given by

$$\hat{n} = \begin{pmatrix} 2/\sqrt{13} \\ 0 \\ 3/\sqrt{13} \end{pmatrix} \quad (61)$$

The angle of rotation is given by the solution to

$$\cos \Phi = -\frac{5}{8} \quad (62)$$

which is about  $129^\circ$ .

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<sup>5</sup>Please refer to the file, `problem4.nb`.



```

In[146]:= Clear["Global`*"];

In[147]:= Mz = {{0, -1, 0}, {1, 0, 0}, {0, 0, 0}};

In[148]:= Rz[x_] = {{Cos[x], Sin[x], 0}, {-Sin[x], Cos[x], 0}, {0, 0, 1}};

In[149]:= Ry[x_] = {{Cos[x], 0, Sin[x]}, {0, 1, 0}, {-Sin[x], 0, Cos[x]}};

In[150]:= Rn = Rz[φ].Ry[θ];

In[151]:= Rn // MatrixForm // FullSimplify
Out[151]/MatrixForm=

$$\begin{pmatrix} \cos[\theta] \cos[\phi] & \sin[\phi] & \cos[\phi] \sin[\theta] \\ -\cos[\theta] \sin[\phi] & \cos[\phi] & -\sin[\theta] \sin[\phi] \\ -\sin[\theta] & 0 & \cos[\theta] \end{pmatrix}$$


In[152]:= R = Rn.MatrixExp[-ϕ * Mz].Transpose[Rn];

In[153]:= R // MatrixForm // FullSimplify
Out[153]/MatrixForm=

$$\begin{pmatrix} \cos[\phi]^2 (\cos[\theta]^2 \cos[\phi] + \sin[\theta]^2) + \cos[\phi] \sin[\phi]^2 & -\sin[\theta]^2 \sin[2\phi] \sin\left[\frac{\phi}{2}\right]^2 + \cos[\theta] \sin[\phi] \cos[\phi] & -\sin[\theta] \sin[\phi] \cos[\phi] \\ -\sin[\theta]^2 \sin[2\phi] \sin\left[\frac{\phi}{2}\right]^2 - \cos[\theta] \sin[\phi] & \cos[\phi]^2 \cos[\phi] + (\cos[\theta]^2 \cos[\phi] + \sin[\theta]^2) \sin[\phi] & \cos[\phi] \sin[\phi] \\ -\sin[\theta] (\cos[\theta] \cos[\phi] (-1 + \cos[\phi]) + \sin[\phi] \sin[\phi]) & \sin[\theta] (\cos[\theta] (-1 + \cos[\phi]) \sin[\phi] - \cos[\phi] \sin[\phi]) & \cos[\theta] \cos[\phi] \end{pmatrix}$$


In[154]:= n = {{Sin[θ] Cos[φ]}, {-Sin[θ] Sin[φ]}, {Cos[θ]}}

Out[154]= {{Cos[φ] Sin[θ]}, {-Sin[θ] Sin[φ]}, {Cos[θ]}}

In[155]:= R.n // MatrixForm // Simplify
Out[155]/MatrixForm=

$$\begin{pmatrix} \cos[\phi] \sin[\theta] \\ -\sin[\theta] \sin[\phi] \\ \cos[\theta] \end{pmatrix}$$


In[156]:= % == n
Out[156]= True

In[157]:= Tr[R] // FullSimplify
Out[157]= 1 + 2 Cos[φ]

```

```
In[123]:= Clear["Global`*"];
```

```
In[124]:= Rx[x_] = {{1, 0, 0}, {0, Cos[x], Sin[x]}, {0, -Sin[x], Cos[x]}};
```

```
In[125]:= Rz[x_] = {{Cos[x], Sin[x], 0}, {-Sin[x], Cos[x], 0}, {0, 0, 1}};
```

```
In[126]:= Rx[θ] // MatrixForm
```

```
Out[126]//MatrixForm=
```

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos[\theta] & \sin[\theta] \\ 0 & -\sin[\theta] & \cos[\theta] \end{pmatrix}$$

```
In[127]:= Rz[φ] // MatrixForm
```

```
Out[127]//MatrixForm=
```

$$\begin{pmatrix} \cos[\phi] & \sin[\phi] & 0 \\ -\sin[\phi] & \cos[\phi] & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

```
In[128]:= R = Rz[π / 3] . Rx[π / 3] . Rz[π / 3];
```

```
In[129]:= R // MatrixForm
```

```
Out[129]//MatrixForm=
```

$$\begin{pmatrix} -\frac{1}{8} & \frac{3\sqrt{3}}{8} & \frac{3}{4} \\ -\frac{3\sqrt{3}}{8} & -\frac{5}{8} & \frac{\sqrt{3}}{4} \\ \frac{3}{4} & -\frac{\sqrt{3}}{4} & \frac{1}{2} \end{pmatrix}$$

```
In[130]:= Tr[R]
```

```
Out[130]= -\frac{1}{4}
```

```
In[136]:= sol = Solve[2 * Cos[φ] + 1 == Tr[R], φ]
```

```
Solve::ifun :
```

```
Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. >>
```

```
Out[136]= {{φ → -ArcCos[-\frac{5}{8}]}, {φ → ArcCos[-\frac{5}{8}]}}
```

```
In[145]:= N[sol[[2]][[1]][[2]], 3] * 180 / π
```

```
Out[145]= 129.
```

```
In[133]:= Eigenvalues[R]
```

```
Out[133]= {\frac{1}{8} (-5 + i \sqrt{39}), \frac{1}{8} (-5 - i \sqrt{39}), 1}
```

```
In[134]:= Eigenvectors[R] // ColumnForm
```

```
Out[134]= {-\frac{3}{2}, -\frac{13 i \sqrt{3} + 3 \sqrt{13}}{2 (-3 i + \sqrt{39})}, 1}
```

$$\left\{-\frac{3}{2}, -\frac{-13 i \sqrt{3} + 3 \sqrt{13}}{2 (3 i + \sqrt{39})}, 1\right\}$$

$$\left\{\frac{2}{3}, 0, 1\right\}$$

```
In[135]:= Transpose[Eigenvalues[R]] // MatrixForm // FullSimplify
```

```
Out[135]//MatrixForm=
```

$$\begin{pmatrix} -\frac{3}{2} & -\frac{3}{2} & \frac{2}{3} \\ -\frac{i\sqrt{13}}{2} & \frac{i\sqrt{13}}{2} & 0 \\ 1 & 1 & 1 \end{pmatrix}$$