Theoretical Dynamics	November 05, 2010

Homework 8

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Problem 1

The effective potential[1] is

$$V_{\text{eff}}(r) = V(r) + \frac{l^2}{2\mu r^2} \tag{1}$$

$$= -\frac{a}{r^k} + \frac{l^2}{2\mu r^2} \tag{2}$$

So,

$$\frac{\partial V_{\text{eff}}}{\partial r} = \frac{ak}{r^{k+1}} - \frac{l^2}{\mu r^3} \tag{3}$$

$$\frac{\partial^2 V_{\text{eff}}}{\partial r^2} = -\frac{ak(k+1)}{r^{k+2}} + \frac{3l^2}{\mu r^4} \tag{4}$$

Therefore,

$$\left.\frac{\partial V_{\text{eff}}}{\partial r}\right|_{r=r_0} = 0 \implies \boxed{r_0 = \left(\frac{ak\mu}{l^2}\right)^{\frac{1}{k-2}}} \tag{5}$$

Also, for $k < 2$, using Mathematica, one can verify that $\left.\frac{\partial^2 V_{\text{eff}}}{\partial r^2}\right|_{r=r_0} > 0$, so $r = r_0$ is a minima of $V_{\text{eff}}(r)$ for $k < 2$.

Part (a)

The condition for a circular orbit is that $E = V_{\text{eff}}(r)$ has only one solution, $r = r_0$ (defined by Eq. (5)), that is,

$$E = V_{\text{eff}}(r_0) \tag{6}$$

where E denotes the total energy. This gives

$$\boxed{E = a\left[\frac{k}{2}\left(\frac{ak\mu}{l^2}\right)^{\frac{k}{k-2}} - \left(\frac{ak\mu}{l^2}\right)^{-\frac{k}{k-2}}\right]} \tag{7}$$

for $k \neq 2$. So, a circular orbit exists for all $k \neq 2$ ($k > 0$). The relation between r_0 and l is

$$\boxed{r_0 = \left(\frac{ak\mu}{l^2}\right)^{\frac{1}{k-2}}} \tag{8}$$

which is meaningful for $k \neq 2$.

[1] In this document, the effective potential is the sum of the actual potential and the centrifugal barrier, i.e. $V_{\text{eff}}(r) = V(r) + V_{cfb}(r) = V(r) + \frac{l^2}{2\mu r^2}$. In Goldstein's notation, $V'(r) = V_{\text{eff}}(r)$.

Part (b)

The equation of motion is

$$\mu\ddot{r} - \frac{l^2}{\mu r^3} = -\frac{ak}{r^{k+1}} \tag{9}$$

Let $r = r_0 + \zeta$. So,

$$\begin{aligned}
\mu\ddot{\zeta} - \frac{l^2}{\mu r_0^3}\left(1 + \frac{\zeta}{r_0}\right)^{-3} &= -\frac{ak}{r_0^{k+1}}\left(1 + \frac{\zeta}{r_0}\right)^{-(k+1)} \\
\implies \mu\ddot{\zeta} - \frac{l^2}{\mu r_0^3}\left(1 - 3\frac{\zeta}{r_0}\right) &= -\frac{ak}{r_0^{k+1}}\left(1 - (k+1)\frac{\zeta}{r_0}\right) \qquad \text{(to first order in } \zeta\text{)} \\
\implies \mu\ddot{\zeta} + \left(\frac{3l^2}{\mu r_0^4} - \frac{k(k+1)a}{r_0^{k+2}}\right)\zeta &= \frac{l^2}{\mu r_0^3} - \frac{ak}{r_0^{k+1}} = 0 \qquad \text{(using Eq. (8))} \\
\implies \ddot{\zeta} + \omega_0^2\zeta &= 0 \qquad (10)
\end{aligned}$$

where the angular frequency ω_0 is given by

$$\boxed{\omega_0 = \sqrt{\frac{3l^2}{\mu^2 r_0^4} - \frac{k(k+1)a}{\mu r_0^{k+2}}}} \tag{11}$$

Note that this linearization of the equation of motion, about the minima of $V_{\text{eff}}(r)$ is equivalent to a harmonic approximation of $V_{\text{eff}}(r)$ itself. Using Mathematica, we can verify that the second derivative of $V_{\text{eff}}(r)$ (which is directly proportional to ω_0^2) at $r = r_0$, is positive for $k < 2$. Therefore, ω_0 so determined is real, for $k < 2$.

The orbits are **stable** if $\partial^2 V_{\text{eff}}(r)/\partial r^2 > 0$ at $r = r_0$, which is true for $k < 2$.

The force corresponding to the given potential is $f(r) = -\partial V/\partial r = ar^{-k-1}$. From Bertrand's Theorem, for the orbits to be closed, we require $-k-1 = -2$ or $-k-1 = 1$, i.e. $k = 1$ or $k = -2$. As $k > 0$, **closed** orbits are possible only if $k = 1$.

Problem 2

The effective potential is

$$\begin{aligned}
V_{\text{eff}}(r) &= V(r) + \frac{l^2}{2\mu r^2} \qquad (12) \\
&= ar^k + \frac{l^2}{2\mu r^2} \qquad (13)
\end{aligned}$$

So,

$$\begin{aligned}
\frac{\partial V_{\text{eff}}}{\partial r} &= akr^{k-1} - \frac{l^2}{\mu r^3} \qquad (14) \\
\frac{\partial^2 V_{\text{eff}}}{\partial r^2} &= ak(k-1)r^{k-2} + \frac{3l^2}{\mu r^4} \qquad (15)
\end{aligned}$$

Therefore,

$$\left.\frac{\partial V_{\text{eff}}}{\partial r}\right|_{r=r_0} = 0 \implies \boxed{r_0 = \left(\frac{ak\mu}{l^2}\right)^{-\frac{1}{k+2}}} \tag{16}$$

Part (a)

The condition for circular orbits to exist is $E = V_{\text{eff}}(r_0)$ where E denotes the total energy, i.e.

$$E = a\left(\frac{ak\mu}{l^2}\right)^{-\frac{k}{k+2}} + \frac{l^2}{2\mu}\left(\frac{ak\mu}{l^2}\right)^{\frac{2}{k+2}} \tag{17}$$

This equation can be satisfied for all k, and hence circular orbits are possible for all values of k ($k > 0$).

Part (b)

The equation of motion is

$$\mu\ddot{r} - \frac{l^2}{\mu r^3} = -akr^{k-1} \tag{18}$$

Let $r = r_0 + \zeta$. So,

$$\begin{aligned}
\mu\ddot{\zeta} - \frac{l^2}{\mu r_0^3}\left(1 + \frac{\zeta}{r_0}\right)^{-3} &= -akr_0^{k-1}\left(1 + \frac{\zeta}{r_0}\right)^{k-1} \\
\implies \mu\ddot{\zeta} - \frac{l^2}{\mu r_0^3}\left(1 - 3\frac{\zeta}{r_0}\right) &= -akr_0^{k-1}\left(1 + (k-1)\frac{\zeta}{r_0}\right) \qquad \text{(to first order in } \zeta\text{)} \\
\implies \mu\ddot{\zeta} + \left(\frac{3l^2}{\mu r_0^4} + ak(k-1)r_0^{k-2}\right)\zeta &= \frac{l^2}{\mu r_0^3} - akr_0^{k-1} = 0 \qquad \text{(from Eq. (16))} \\
\implies \ddot{\zeta} + \omega_0^2\zeta &= 0 \qquad (19)
\end{aligned}$$

where the oscillation frequency ω_0 is given by

$$\boxed{\omega_0 = \sqrt{\frac{3l^2}{\mu r_0^4} + ak(k-1)r_0^{k-2}}} \tag{20}$$

The force corresponding to the given potential is $f(r) = -akr^{k-1}$. From Bertrand's Theorem, closed orbits exist for $k - 1 = -2$ and $k - 1 = 1$, i.e. for $k = -1$ and $k = 2$. As $k > 0$, **closed** orbits are possible only for $k = 2$ (which corresponds to Hooke's law). These orbits are necessarily stable, as $\partial^2 V_{\text{eff}}/\partial r^2 > 0$ for these orbits, at $r = r_0$.

Problem 3

Part (a)

$$\begin{aligned}\{\boldsymbol{A}, H\} &= \left\{\boldsymbol{p}\times\boldsymbol{L} - k\mu\hat{\boldsymbol{r}}, \frac{\boldsymbol{p}^2}{2\mu} - \frac{k}{r}\right\} \\ &= \left\{\boldsymbol{p}\times\boldsymbol{L}, \frac{\boldsymbol{p}^2}{2\mu}\right\} - \left\{\boldsymbol{p}\times\boldsymbol{L}, \frac{k}{r}\right\} - \left\{k\mu\hat{\boldsymbol{r}}, \frac{\boldsymbol{p}^2}{2\mu}\right\} + \left\{k\mu\hat{\boldsymbol{r}}, \frac{k}{r}\right\} \end{aligned} \tag{21}$$

We can consider each term separately.

$$\begin{aligned}\left\{(\boldsymbol{p}\times\boldsymbol{L})_i, \frac{\boldsymbol{p}^2}{2\mu}\right\} &= \frac{1}{2\mu}\{\epsilon_{ijk}p_jL_k, p_mp_m\} \\ &= \frac{\epsilon_{ijk}}{\mu}\{p_jL_k, p_m\}p_m \\ &= \frac{\epsilon_{ijk}}{\mu}\left[p_j\{L_k, p_m\} + \{p_j, p_m\}L_k\right]p_m \\ &= \frac{\epsilon_{ijk}}{\mu}\{\epsilon_{krs}\mu x_rp_s, p_m\}p_jp_m \\ &= \epsilon_{ijk}\epsilon_{krs}\left[\{x_r, p_m\}p_s + x_r\{p_s, p_m\}\right]p_jp_m \\ &= \epsilon_{ijk}\epsilon_{krs}\delta_{rm}p_sp_jp_m \\ &= \epsilon_{kij}\delta_{kms}p_sp_jp_m \\ &= (\delta_{im}\delta_{js} - \delta_{is}\delta_{jm})p_sp_jp_m \\ &= p_ip_sp_s - p_ip_mp_m \\ &= 0 \end{aligned} \tag{22}$$

$$\begin{aligned}\left\{(\boldsymbol{p}\times\boldsymbol{L})_i, \frac{k}{r}\right\} &= \left\{\epsilon_{ijk}p_jL_k, \frac{k}{r}\right\} \\ &= k\epsilon_{ijk}\left\{p_jL_k, \frac{1}{r}\right\} \\ &= k\epsilon_{ijk}\left[\left\{p_j, \frac{1}{r}\right\}L_k + p_j\left\{L_k, \frac{1}{r}\right\}\right] \\ &= k\epsilon_{ijk}\left[\frac{x_j}{r^3}L_k + p_j\left\{\epsilon_{kmn}x_mp_n, \frac{1}{r}\right\}\right] \qquad \left(\text{using } \left\{p_n, \tfrac{1}{r}\right\} = \tfrac{x_n}{r^3}\right) \\ &= k\epsilon_{ijk}\left[\frac{x_j}{r^3}L_k + \epsilon_{kmn}p_jx_m\left\{p_n, \frac{1}{r}\right\}\right] \\ &= k\epsilon_{ijk}\left[\frac{x_j}{r^3}L_k + \epsilon_{kmn}p_jx_m\frac{x_n}{r}\right] \\ &= k\epsilon_{ijk}\left[\frac{x_j}{r^3}\epsilon_{kmn}x_mp_n\right] \qquad (\text{due to antisymmetry of } \epsilon_{ijk}) \\ &= \frac{k}{r^3}(\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})x_jx_mp_n \\ &= \frac{k}{r^3}(x_ix_np_n - x_mx_mp_i) \\ &= k\frac{x_ix_mp_m}{r^3} - k\frac{p_i}{r} \end{aligned} \tag{23}$$

$$
\begin{aligned}
k\left\{\hat{\boldsymbol{r}}_i, \frac{\boldsymbol{p}^2}{2}\right\} &= k\left\{\frac{x_i}{r}, p_m\right\} p_m \\
&= k\frac{1}{r}\{x_i, p_m\} p_m + x_i\left\{\frac{1}{r}, p_m\right\} p_m \\
&= k\frac{1}{r}\delta_{im} p_m - x_i \frac{x_m}{r^3} p_m \qquad \left(\text{using } \left\{\tfrac{1}{r}, p_n\right\} = -\tfrac{x_n}{r^3}\right) \\
&= k\frac{p_i}{r} - k\frac{x_i x_m p_m}{r^3} \qquad (24)
\end{aligned}
$$

$$
\begin{aligned}
\left\{\hat{\boldsymbol{r}}_i, \frac{1}{r^2}\right\} &= \left\{\frac{x_i}{r}, \frac{1}{r^2}\right\} \\
&= 0 \qquad (25)
\end{aligned}
$$

So, from Eqs. (21) through (25) we get

$$\{A_i, H\} = 0 \qquad (26)$$

for each $i = 1, 2, 3$, which implies

$$\boxed{\{\boldsymbol{A}, H\} = 0} \qquad (27)$$

Part (b)

The angular momentum vector $\boldsymbol{L}$ is perpendicular to the plane of the orbit:

$$
\begin{aligned}
\boldsymbol{L} \cdot \boldsymbol{r} &= L_i x_i \\
&= \epsilon_{ijk} x_i x_j p_k \\
&= 0 \qquad (\text{due to antisymmetry of } \epsilon_{ijk})
\end{aligned}
$$

and

$$
\begin{aligned}
\boldsymbol{L} \cdot \boldsymbol{p} &= L_i p_i \\
&= \epsilon_{ijk} x_i p_j p_k \\
&= 0 \qquad (\text{due to antisymmetry of } \epsilon_{ijk})
\end{aligned}
$$

So,

$$
\begin{aligned}
\boldsymbol{L} \cdot \boldsymbol{A} &= \boldsymbol{L} \cdot (\boldsymbol{p} \times \boldsymbol{L} - k\mu\hat{\boldsymbol{r}}) \\
&= \boldsymbol{L} \cdot (\boldsymbol{p} \times \boldsymbol{L}) \\
&= \boldsymbol{L} \cdot (\boldsymbol{p} \times (\boldsymbol{r} \times \boldsymbol{p})) \\
&= \boldsymbol{L} \cdot ((\boldsymbol{p} \cdot \boldsymbol{p})\boldsymbol{r} - (\boldsymbol{r} \cdot \boldsymbol{p})\boldsymbol{p}) \\
&= (\boldsymbol{p} \cdot \boldsymbol{p})(\boldsymbol{L} \cdot \boldsymbol{r}) - (\boldsymbol{r} \cdot \boldsymbol{p})(\boldsymbol{L} \cdot \boldsymbol{p}) \\
&= 0 \qquad (28)
\end{aligned}
$$

As $\boldsymbol{A}$ is perpendicular to $\boldsymbol{L}$ (which is orthogonal to the plane of the orbit), it must lie in the plane of the orbit.

Problem 4

$$\begin{aligned}\{A_i, L_j\} &= \left\{(\boldsymbol{p}\times\boldsymbol{L})_i - \frac{k\mu x_i}{r}, L_j\right\} \\ &= \{(\boldsymbol{p}\times\boldsymbol{L})_i, L_j\} - k\mu\left\{\frac{x_i}{r}, L_j\right\} \qquad (29)\end{aligned}$$

The first term is,

$$\begin{aligned}\{(\boldsymbol{p}\times\boldsymbol{L})_i, L_j\} &= \{\epsilon_{imn} p_m L_n, L_j\} \\ &= \epsilon_{imn}[p_m\{L_n, L_j\} + \{p_m, L_j\}L_n] \qquad (30)\end{aligned}$$

Now,

$$\begin{aligned}\{L_n, L_j\} &= \epsilon_{nrs}\epsilon_{jpq}\{x_r p_s, x_p p_q\} \\ &= \epsilon_{nrs}\epsilon_{jpq}(\delta_{rq} x_p p_s - \delta_{ps} x_r p_q) \\ &= \epsilon_{nqs}\epsilon_{jpq} x_p p_s - \epsilon_{nrs}\epsilon_{jsq} x_r p_q \\ &= \epsilon_{snr}\epsilon_{sjq} x_r p_q - \epsilon_{qns}\epsilon_{qjp} x_p p_s \\ &= (\delta_{nj}\delta_{rq} - \delta_{nq}\delta_{rj}) x_r p_q - (\delta_{nj}\delta_{sp} - \delta_{np}\delta_{sj}) x_p p_s \\ &= x_n p_j - x_j p_n \\ &= \epsilon_{njk} L_k \qquad (31)\end{aligned}$$

and

$$\begin{aligned}\{p_m, L_n\} &= \epsilon_{nrs}\{p_m, x_r p_s\} \\ &= -\epsilon_{nrs}\delta_{rm} p_s \\ &= \epsilon_{mns} p_s \qquad (32)\end{aligned}$$

Also,

$$\begin{aligned}\left\{\frac{x_i}{r}, L_j\right\} &= \epsilon_{jmn}\left\{\frac{x_i}{r}, x_m p_n\right\} \\ &= \epsilon_{jmn}\left\{\frac{x_i}{r}, p_n\right\} x_m \\ &= \epsilon_{jmn}\left[\frac{1}{r}\{x_i, p_n\} + x_i\left\{\frac{1}{r}, p_n\right\}\right] x_m \\ &= \epsilon_{jmn}\left[\frac{\delta_{in}}{r} - \frac{x_i x_n}{r^3}\right] x_m \qquad \left(\text{using } \left\{\tfrac{1}{r}, p_n\right\} = -\tfrac{x_n}{r^3}\right) \\ &= \frac{\epsilon_{ijm} x_m}{r} \qquad (33)\end{aligned}$$

So, from Eqs. (30), (31) and (32) we have

$$\begin{aligned}\{(\boldsymbol{p}\times\boldsymbol{L})_i, L_j\} &= \epsilon_{imn}(\epsilon_{njk} p_m L_k + \epsilon_{mjk} p_k L_n) \\ &= \epsilon_{nim}\epsilon_{njk} p_m L_k - \epsilon_{min}\epsilon_{mjk} p_k L_n \\ &= (\delta_{ij}\delta_{mk} - \delta_{ik}\delta_{mj}) p_m L_k - (\delta_{ij}\delta_{nk} - \delta_{ik}\delta_{nj}) p_k L_n \\ &= p_i L_j - p_j L_i \qquad (34)\end{aligned}$$

And from Eqs. (30), (33) and (34) we get

$$\{A_i, L_j\} = p_i L_j - p_j L_i - \frac{k\mu\epsilon_{ijm}x_m}{r} \tag{35}$$

But

$$\begin{aligned}
A_k &= \epsilon_{kmn}p_m L_n - k\mu\frac{x_k}{r} \\
\Longrightarrow \epsilon_{ijk}A_k &= \epsilon_{ijk}\epsilon_{kmn}p_m L_n - \frac{k\mu\epsilon_{ijk}x_k}{r} \\
\Longrightarrow \epsilon_{ijk}A_k &= (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})p_m L_n - \frac{k\mu\epsilon_{ijm}x_m}{r} \\
\Longrightarrow \epsilon_{ijk}A_k &= p_i L_j - p_j L_i - \frac{k\mu\epsilon_{ijm}x_m}{r} \qquad (36)
\end{aligned}$$

So, from Eqs. (35) and (36), we have

$$\boxed{\{A_i, L_j\} = \epsilon_{ijk}A_k} \tag{37}$$

In particular, this implies

$$\{A_x, L_y\} = A_z \tag{38}$$

Next, we compute $\{A_i, A_j\}$:

$$\begin{aligned}
\{A_i, A_j\} &= \{(\boldsymbol{p}\times\boldsymbol{L})_i, (\boldsymbol{p}\times\boldsymbol{L})_j\} - \mu k\{(\boldsymbol{p}\times\boldsymbol{L})_i, \hat{\boldsymbol{r}}_j\} - \mu k\{\hat{\boldsymbol{r}}_i, (\boldsymbol{p}\times\boldsymbol{L})_j\} + \mu^2 k^2\{\hat{\boldsymbol{r}}_i, \hat{\boldsymbol{r}}_j\} \\
&= \{(\boldsymbol{p}\times\boldsymbol{L})_i, (\boldsymbol{p}\times\boldsymbol{L})_j\} - \mu k\{(\boldsymbol{p}\times\boldsymbol{L})_i, \hat{\boldsymbol{r}}_j\} - \mu k\{\hat{\boldsymbol{r}}_i, (\boldsymbol{p}\times\boldsymbol{L})_j\} \qquad (39)
\end{aligned}$$

Now,

$$\begin{aligned}
\{(\boldsymbol{p}\times\boldsymbol{L})_i, (\boldsymbol{p}\times\boldsymbol{L})_j\} &= \epsilon_{jlm}\{(\boldsymbol{p}\times\boldsymbol{L})_i, p_l L_m\} \\
&= \epsilon_{jlm}\left[\{(\boldsymbol{p}\times\boldsymbol{L})_i, p_l\}L_m + p_l\{(\boldsymbol{p}\times\boldsymbol{L})_i, L_m\}\right] \\
&= \epsilon_{jlm}\left[\epsilon_{ijk}p_j\{L_k, p_l\}L_m + p_l(p_i L_m - p_m L_i)\right] \qquad \text{(using Eq. (34))} \\
&= \epsilon_{jlm}\left[\epsilon_{ijk}p_j\epsilon_{kls}p_s L_m + p_l(p_i L_m - p_m L_i)\right] \qquad \text{(using Eq. (32))} \\
&= \epsilon_{jlm}\left[(\delta_{il}\delta_{js} - \delta_{is}\delta_{jl})\right]p_j p_s + \epsilon_{jlm}p_l(p_i L_m - p_m L_i) \\
&= \epsilon_{jim}p^2 L_m - p_i\epsilon_{jlm}p_l L_m + p_i\epsilon_{jlm}p_l L_m - \epsilon_{jlm}p_l p_m L_i \\
&= -\epsilon_{ijm}p^2 L_m \qquad (40)
\end{aligned}$$

and

$$\begin{aligned}
\{(\boldsymbol{p}\times\boldsymbol{L})_i, \hat{r}_j\} &= \left\{(\boldsymbol{p}\times\boldsymbol{L})_i, \frac{x_j}{r}\right\} \\
&= \left\{(\boldsymbol{p}\times\boldsymbol{L})_i, \frac{1}{r}\right\} x_j + \{(\boldsymbol{p}\times\boldsymbol{L})_i, x_j\}\frac{1}{r} \\
&= \left(\frac{x_i x_m p_m}{r^3} - \frac{p_i}{r}\right) x_j + \{(\boldsymbol{p}\times\boldsymbol{L})_i, x_j\}\frac{1}{r} \qquad \text{(from Eq. (23))} \\
&= \left(\frac{x_i x_m p_m}{r^3} - \frac{p_i}{r}\right) x_j + \epsilon_{ilm}\left[\{p_l, x_j\}L_m + p_l\{L_m, x_j\}\right]\frac{1}{r} \\
&= \left(\frac{x_i x_m p_m}{r^3} - \frac{p_i}{r}\right) x_j + \epsilon_{ilm}\left[-\delta_{jl}L_m + p_l\epsilon_{mns}x_n\{p_s, x_j\}\right]\frac{1}{r} \\
&= \left(\frac{x_i x_m p_m}{r^3} - \frac{p_i}{r}\right) x_j + \frac{1}{r}\left(-\epsilon_{ijm}L_m - \epsilon_{ilm}\epsilon_{mns}p_l x_n\delta_{js}\right) \\
&= \left(\frac{x_i x_m p_m}{r^3} - \frac{p_i}{r}\right) x_j + \frac{1}{r}\left(-\epsilon_{ijm}L_m - (\delta_{in}\delta_{lj} - \delta_{ij}\delta_{ln})p_l x_n\right) \\
&= \left(\frac{x_i x_m p_m}{r^3} - \frac{p_i}{r}\right) x_j + \frac{\delta_{ij}p_l x_l - x_i p_j - \epsilon_{ijm}L_m}{r} \qquad (41)
\end{aligned}$$

From Eqs. (39) through (41), we have

$$\begin{aligned}
\{A_i, A_j\} &= \{(\boldsymbol{p}\times\boldsymbol{L})_i, (\boldsymbol{p}\times\boldsymbol{L})_j\} - \mu k\{(\boldsymbol{p}\times\boldsymbol{L})_i, \hat{\boldsymbol{r}}_j\} - \mu k\{\hat{\boldsymbol{r}}_i, (\boldsymbol{p}\times\boldsymbol{L})_j\} \\
&= -\epsilon_{ijm}p^2 L_m + \mu k\left[\left(\frac{p_i}{r} - \frac{x_i x_m p_m}{r^3}\right) x_j + \frac{x_i p_j + \epsilon_{ijm}L_m - \delta_{ij}p_l x_l}{r}\right] \\
&\quad + \mu k\left[\left(\frac{x_j x_m p_m}{r^3} - \frac{p_j}{r}\right) x_i + \frac{\delta_{ij}p_l x_l - x_j p_i + \epsilon_{ijm}L_m}{r}\right] \\
&= -p^2\epsilon_{ijm}L_m + \frac{2\mu k\epsilon_{ijm}L_m}{r} \\
&= -2\mu\epsilon_{ijm}\left(\frac{p^2}{2\mu} - \frac{k}{r}\right)L_m \\
&= -2\mu H\epsilon_{ijm}L_m \qquad (42) \\
&= -2\mu E\epsilon_{ijm}L_m \qquad \text{(as } E = H\text{)} \qquad (43)
\end{aligned}$$

So,

$$\boxed{\{A_i, A_j\} = -2\mu E\epsilon_{ijk}A_k} \qquad (44)$$

In particular,

$$\{A_x, A_y\} = -2\mu E A_z \qquad (45)$$

Problem 5

Let T and V denote the potential and kinetic energies, i.e.

$$\begin{aligned}
T &= \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r\dot{\theta}^2 \qquad (46) \\
&= \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\frac{l^2}{\mu r^2} \qquad (47) \\
V &= -\frac{k}{r} \qquad (48)
\end{aligned}$$

From the virial theorem, we have

$$\overline{T} = -\frac{1}{2}\overline{V} \tag{49}$$

Also, $E = T + V$, where E is the total energy ($E < 0$ for an elliptic orbit). So, $\overline{E} = E = \overline{T} + \overline{V} = \overline{V}/2 = -\overline{T}$. So,

$$\begin{aligned} \frac{1}{2}\overline{\left(-\frac{k}{r}\right)} &= E && (50)\\ -\left[\frac{1}{2}\mu\overline{\dot{r}^2} + \frac{l^2}{2\mu}\overline{\frac{1}{r^2}}\right] &= E && (51) \end{aligned}$$

The first equation gives

$$\boxed{\overline{r^{-1}} = -\frac{2E}{k}} \tag{52}$$

Now,

$$\begin{aligned} \overline{r^{-2}} &= \frac{1}{2T}\int_{-T}^{T}\frac{1}{r^2}dt \\ &= \frac{1}{T}\int_0^T \frac{1}{r^2}dt \qquad \text{(taking } T \text{ to be one period)} \\ &= \frac{1}{T}\int_0^{2\pi}\frac{1}{r^2}\frac{dt}{d\theta}d\theta \\ &= \frac{1}{T}\int_0^{2\pi}\frac{1}{r^2}\frac{\mu r^2}{l}d\theta \qquad \text{(using } l = \mu r^2\dot{\theta}\text{)} \\ &= \frac{2\pi\mu}{lT} \\ &= \frac{2\pi\mu}{2\pi l a^{3/2}\sqrt{\frac{\mu}{k}}} \qquad \text{(using } T = 2\pi a^{3/2}\sqrt{\tfrac{\mu}{k}}\text{)} \\ &= \frac{1}{l}\sqrt{\frac{\mu k}{a^3}} \qquad \text{(where } a \text{ is the semimajor axis)} && (53) \end{aligned}$$

So,

$$\boxed{\overline{r^{-2}} = \frac{1}{l}\sqrt{\frac{\mu k}{a^3}}} \tag{54}$$

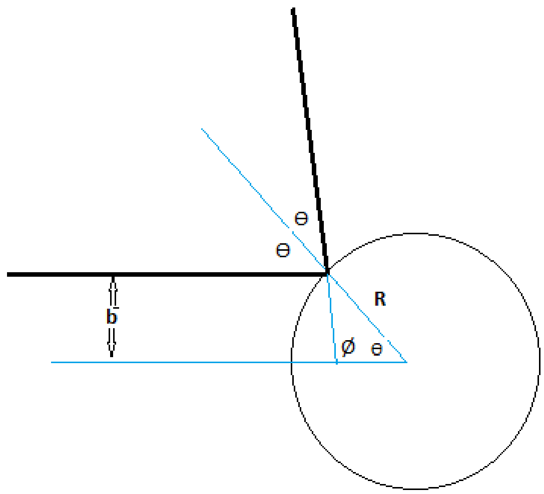
From Eq. (51),

$$\begin{aligned} \overline{\dot{r}^2} &= \frac{2}{\mu}\left[-E - \frac{l^2}{2\mu}\overline{r^{-2}}\right] \\ &= -\frac{2E}{\mu} - \frac{l^2}{\mu^2}\left(\frac{1}{l}\sqrt{\frac{\mu k}{a^3}}\right) \\ &= -\frac{2E}{\mu} - l\sqrt{\frac{k}{\mu^3 a^3}} && (55) \end{aligned}$$

So,

$$\boxed{\overline{\dot{r}^2} = -\frac{2E}{\mu} - l\sqrt{\frac{k}{\mu^3 a^3}}} \tag{56}$$

Problem 6



Let I denote the intensity of the beam of incident particles. So, the number of particles scattered into a solid angle $d\Omega$ is $I \times 2\pi b db$ where b denotes the impact parameter. From the geometry of the figure,

$$\text{Impact parameter} = b = R\sin\theta \tag{57}$$
$$\text{Scattering angle} = 2\theta \tag{58}$$

So,

$$d\Omega = 2\pi\sin(2\theta)d(2\theta) \tag{59}$$

The differential scattering cross-section is given by

$$\begin{aligned}\frac{d\sigma}{d\Omega}d\Omega &= \frac{\text{no of particles scattered into a solid angle } d\Omega}{\text{intensity of incident particles}} \\ &= \frac{I \times 2\pi b db}{I}\end{aligned} \tag{60}$$

that is

$$\frac{d\sigma}{d\Omega} \times (4\pi\sin(2\theta)d\theta) = 2\pi(R\sin\theta)(R\cos\theta)d\theta \tag{61}$$

So, the differential scattering cross-section is

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{R^2}{4}} \tag{62}$$

Problem 7

From the orbit equation,

$$\begin{aligned}
\frac{dr}{d\theta} &= \frac{\mu r^2}{l}\sqrt{\frac{2}{\mu}\left(E - \frac{\alpha}{r^2} - \frac{l^2}{2\mu r^2}\right)} \\
&= \frac{\mu r^2}{b\sqrt{2\mu E}}\sqrt{\frac{2}{\mu}\left(E - \frac{\alpha}{r^2} - \frac{2\mu E b^2}{2\mu r^2}\right)} \\
&= \frac{r^2}{b}\sqrt{1 - \left(\frac{\alpha}{E} + b^2\right)\frac{1}{r^2}}
\end{aligned}$$

So, $\frac{dr}{d\theta} = 0$ for $r = 0, \sqrt{\frac{\alpha}{E} + b^2}$. Thus, the distance of closest approach is

$$r_{min} = \sqrt{\frac{\alpha}{E} + b^2} \tag{63}$$

The scattering angle is given by

$$\begin{aligned}
\Theta(b) &= \pi - 2\int_{r_{min}}^{\infty} \frac{b dr}{r\sqrt{r^2\left(1 - \frac{V(r)}{E}\right) - b^2}} \\
&= \pi - 2\int_{\sqrt{\frac{\alpha}{E}+b^2}}^{\infty} \frac{b\,dr}{r^2\left(1 - \left(\frac{\alpha}{E} + b^2\right)\frac{1}{r^2}\right)^{1/2}}
\end{aligned} \tag{64}$$

Let $r = \sqrt{\frac{\alpha}{E} + b^2}\csc\varphi$. Then, $dr = -\sqrt{\frac{\alpha}{E} + b^2}\csc\varphi\cot\varphi$. Substituting into the integral in Eq. (64), this gives

$$\begin{aligned}
\Theta(b) &= \pi - 2\int_{\pi/2}^{0} \frac{b(-\sqrt{\frac{\alpha}{E} + b^2}\csc\varphi\cot\varphi)}{\left(\frac{\alpha}{E} + b^2\right)\cos\varphi} d\varphi \\
&= \pi - \frac{2b}{\sqrt{\frac{\alpha}{E} + b^2}}\frac{\pi}{2} \\
&= \pi - \frac{b\pi}{\sqrt{\frac{\alpha}{E} + b^2}}
\end{aligned} \tag{65}$$

Rearranging this equation and squaring both sides, we get

$$\begin{aligned}
\frac{b^2\pi^2}{\frac{\alpha}{E} + b^2} &= (\pi - \Theta)^2 \\
\Longrightarrow b^2(\pi^2 - (\pi - \theta)^2) &= (\pi - \Theta)^2\frac{\alpha}{E} \\
\Longrightarrow b &= \sqrt{\frac{\alpha}{E}}\frac{\pi - \Theta}{\sqrt{2\pi\Theta - \Theta^2}}
\end{aligned} \tag{66}$$

Therefore,

$$\begin{aligned}
\frac{db}{d\Theta} &= \sqrt{\frac{\alpha}{E}}\left[-\frac{(2\pi - 2\Theta)(\pi - \Theta)}{2(2\pi\Theta - \Theta^2)^{3/2}} - \frac{1}{\sqrt{2\pi\Theta - \Theta^2}}\right] \\
&= -\sqrt{\frac{\alpha}{E}}\frac{\pi^2}{(2\pi\Theta - \Theta^2)^{3/2}}
\end{aligned} \tag{67}$$

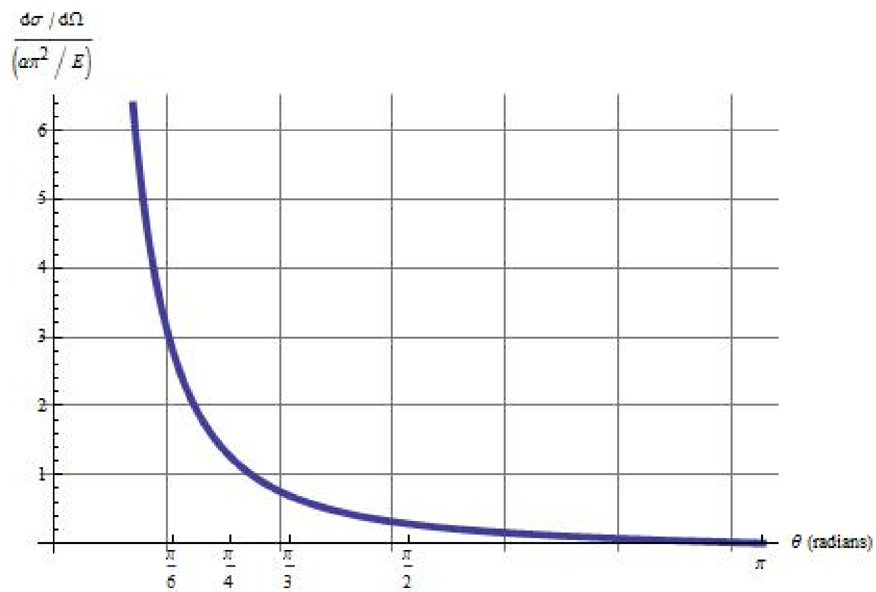
The differential scattering cross-section is given by

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\Theta}\left|\frac{db}{d\Theta}\right| \tag{68}$$
$$= \sqrt{\frac{\alpha}{E}}\frac{\pi-\Theta}{\sqrt{2\pi\Theta-\Theta^2}}\left|-\sqrt{\frac{\alpha}{E}}\frac{\pi^2}{(2\pi\Theta-\Theta^2)^{3/2}}\right|$$

which simplifies to

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{\alpha\pi^2}{E}\left[\frac{|\pi-\Theta|}{\Theta^2(2\pi-\Theta)^2|\sin\Theta|}\right]} \tag{69}$$

A plot of the scaled differential scattering cross-section is shown below.



In[1]:=
```
Clear["Global*`"];
```

In[2]:=
```
Veff[r_] = - a/(r^k) + 1/2 l^2/(μ r^2)
```

Out[2]= $\frac{l^2}{2 r^2 \mu} - a r^{-k}$

In[3]:=
```
D[Veff[r], r] // FullSimplify
```

Out[3]= $a k r^{-k-1} - \frac{l^2}{r^3 \mu}$

In[4]:=
```
D[%, r] // FullSimplify
```

Out[4]= $\frac{3 l^2}{r^4 \mu} - a k (k+1) r^{-k-2}$

In[5]:=
```
Solve[D[Veff[r], r] == 0, r]
```

Solve::ifun :
Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. ≫

Out[5]= $\left\{\left\{r \to \left(\frac{a k \mu}{l^2}\right)^{\frac{1}{k-2}}\right\}\right\}$

In[6]:=
```
r0[k_] = (a * k * μ / l^2)^(1 / (k - 2))
```

Out[6]= $\left(\frac{a k \mu}{l^2}\right)^{\frac{1}{k-2}}$

In[7]:=
```
D[Veff[r], r] /. {r → r0[k]} // FullSimplify
```

Out[7]= $a k \left(\left(\frac{a k \mu}{l^2}\right)^{\frac{1}{k-2}}\right)^{-k-1} - a k \left(\frac{a k \mu}{l^2}\right)^{\frac{k+1}{2-k}}$

In[8]:=
```
D[D[Veff[r], r], r]
```

Out[8]= $a(-k-1) k r^{-k-2} + \frac{3 l^2}{r^4 \mu}$

In[9]:=
```
secondderivative[k_] = D[D[Veff[r], r], r] /. {r → r0[k]} // FullSimplify
```

Out[9]= $a(-k-1) k \left(\left(\frac{a k \mu}{l^2}\right)^{\frac{1}{k-2}}\right)^{-k-2} + 3 a k \left(\frac{a k \mu}{l^2}\right)^{\frac{k+2}{2-k}}$

In[10]:=
```
Assuming[a > 0 && μ > 0 && l > 0, secondderivative[1.5] // FullSimplify]
```

Out[10]= $12.8145\, a \left(\frac{a \mu}{l^2}\right)^{7.}$

In[11]:=
```
Assuming[a > 0 && μ > 0 && l > 0, secondderivative[1.9] // FullSimplify]
```

Out[11]= $1.41301 \times 10^{10}\, a \left(\frac{a \mu}{l^2}\right)^{39.}$

In[12]:=
```
Assuming[a > 0 && μ > 0 && l > 0, secondderivative[1.999] // FullSimplify]
```

Out[12]= $1.78310317207\times 10^{1200}\, a \left(\frac{a\,\mu}{l^2}\right)^{3999.}$

In[13]:=
```
Assuming[a > 0 && μ > 0 && l > 0, secondderivative[2.00001] // FullSimplify]
```

Out[13]= $-\frac{1.3587776791\times 10^{-120\,418}\, a}{\left(\frac{a\,\mu}{l^2}\right)^{400\,001.}}$

In[14]:=
```
Assuming[a > 0 && μ > 0 && l > 0, secondderivative[2.2] // FullSimplify]
```

Out[14]= $-\frac{2.83515\times 10^{-8}\, a}{\left(\frac{a\,\mu}{l^2}\right)^{21.}}$

So, the second derivative of the effective potential is positive at $r_0$only if k < 2. Therefore, stable orbits are possible only for k < 2.

Linearization

In[15]:=
```
y = r0 + x
```

Out[15]= $r_0 + x$

In[16]:=
```
eq = μ r''[t] - l^2/(μ r^3) + a k/(r^(k + 1)) == 0
```

Out[16]= $a\,k\,r^{-k-1} - \frac{l^2}{r^3\,\mu} + \mu\,r''(t) = 0$

In[17]:=
```
Series[eq /. {r → α + z}, {z, 0, 1}]
```

Out[17]= $\mu\,(z+\alpha)''(t) + \left(\left(a\,k\,\alpha^{-k-1} - \frac{l^2}{\alpha^3\,\mu}\right) + \left(-a\,k^2\,\alpha^{-k-2} - a\,k\,\alpha^{-k-2} + \frac{3\,l^2}{\mu\,\alpha^4}\right) z + O\left(z^2\right)\right) = 0$