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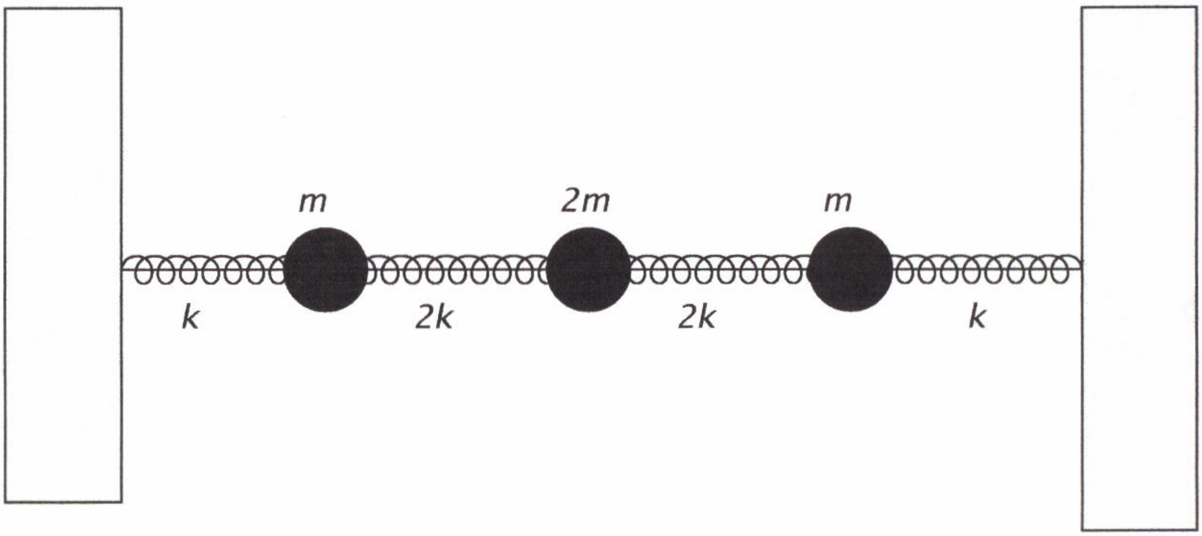
Physics 601 Homework 7---Due Friday October 29

40/40

You may find Mathematica very helpful for some of these problems!!

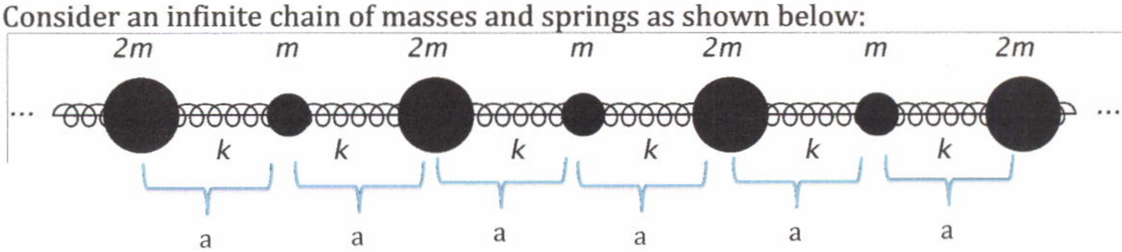
1. Three beads are free to move along a wire. They are connected two immovable walls by 4 springs. The masses and spring constants are as indicated on the figure.



   a. Find the normal modes and their associated frequencies by finding the eigenvectors and eigenvalues of the appropriate matrix.
   b. Verify that the modes are orthogonal with respect to the mass matrix.
   c. Suppose at t=0 all three particles are in their equilibrium positions with the two particle on the end at rest and the one in the middle moving with velocity v. (This can happens as a result of an impulse acting on the middle particle). Find the motion of the three particles.

2. Consider the system described in the previous problem. Suppose that all the particles begin at rest in equilibrium a force given by $f(t) = f_0 \sinh(t/T)\theta(t)$ acts on the middle particle. Use Green's functions to determine the motion of the three particles.

3. Consider an infinite chain of masses and springs as shown below:



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LIMITED-LIABILITY COMPANIES (LLCs)

It may take some searching, but there's a lid for every pot.

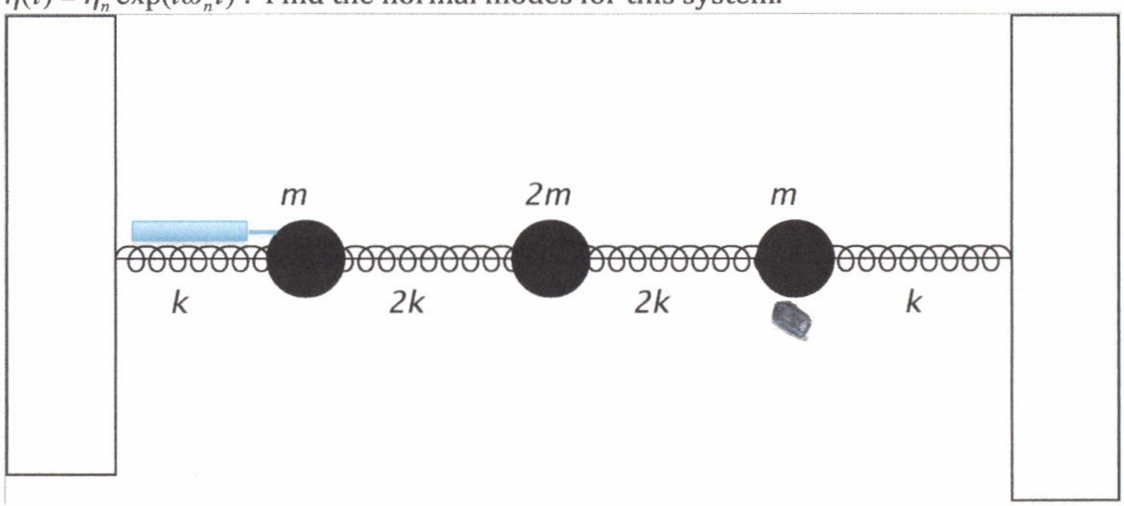
—ANONYMOUS

Mary L——was a schoolteacher who lived with her husband and four children near Seattle, Washington. In the early 1990s, Mary purchased a gray Volkswagen Fox. In all innocence, she titled the car in her own name. Not in her wildest dreams—or nightmares!—could Mary have foreseen the bitter consequences of driving a car with plates that would show up in the computer with her first name, middle initial, and last name.

Years passed. Unexpectedly, Mary fell in love with one of her students, a thirteen-year-old boy. Despite being married, she was unable to control her emotions. One thing led to another and Mary found herself pregnant. In 1977, the affair became known. She was arrested, jailed, and sentenced to eighty-nine months in prison. Given her lack of a criminal record and the fact she posed no threat to society, Mary was released on parole. One of the

The masses alternate between two types, one twice as heavy as the other. The springs connecting the masses are all equal and in equilibrium the masses are all a distance a apart. The masses are constrained to move longitudinally. Find the dispersion relation relating frequency to wave number.

4. Three beads are free to move along a wire. They are connected two immovable walls by 4 springs. The masses and spring constants are as indicated on the figure. The third particle ~~is charged and~~ has a mass q. The first particle is attached to a shock absorber with a force given by $F = -m\omega_0 \dot{x}$ where $\omega_0^2 = \frac{k}{m}$. Using the phase-space formalism derived in class: $\vec{\eta} = \begin{pmatrix} \vec{q} \\ \vec{p} \end{pmatrix}$ $\vec{\vec{W}} = \begin{pmatrix} 0 & (\vec{\vec{T}})^{-1} \\ -\vec{\vec{V}} & -\vec{\vec{F}}(\vec{\vec{T}})^{-1} \end{pmatrix}$ with the equation of motion case given by $\dot{\vec{\eta}} = \vec{\vec{W}}\vec{\eta}$; normal modes are of the form $\vec{\eta}(t) = \vec{\eta}_n \exp(i\omega_n t)$. Find the normal modes for this system.



5. Suppose the system described in the problem above has the middle particle charged (with charge q) and is place in an electric field, oriented to the right with a magnitude of $E = E_0 \cos(2\omega_0 t)$. Find the positions of each mass as a function of time, assuming that the system has reached steady state.

6. Consider a driven damped system of oscillators problem using the phase-space formalism derived in class: $\vec{\eta} = \begin{pmatrix} \vec{q} \\ \vec{p} \end{pmatrix}$ $\vec{\vec{W}} = \begin{pmatrix} 0 & (\vec{\vec{T}})^{-1} \\ -\vec{\vec{V}} & -\vec{\vec{F}}(\vec{\vec{T}})^{-1} \end{pmatrix}$ with the equation of motion for the undriven case given by $\dot{\vec{\eta}} = \vec{\vec{W}}\vec{\eta}$; normal modes are of the form $\vec{\eta}(t) = \vec{\eta}_n \exp(i\omega_n t)$ and satisfy the eigenvalue equation $i\omega_n \vec{\eta}_n = \vec{\vec{W}}\vec{\eta}_n$. Now the driven oscillator of the form $\dot{\vec{\eta}} = \vec{\vec{W}}\vec{\eta} + \vec{f}(t)$ with $\vec{f}(t) = \begin{pmatrix} 0 \\ \vec{F}(t) \end{pmatrix}$. The Green's

conditions of the parole was that she would not contact her young lover without permission from the authorities.

Mary carried a pager. The father of her future child, unable—or unwilling—to stay away from Mary, sent her a page with the number of a pay phone. She called him back. That same evening she picked him up in her little gray Volkswagen Fox.

They went to a late movie, *Wag the Dog.* Then Mary parked her Volkswagen along a street near her home and they talked into the early-morning hours. Mary had already packed and had hidden money and her passport in the car. They apparently made plans to flee Seattle together and to make a new life far, far away.

At that point, was there any obstacle to their plans? The parole board had no idea that Mary was violating parole. She was not wanted by the police. No one else knew of their plans to flee. Nor was society in danger.

[Imagine background music to the movie *Jaws.*]

At 2:45 A.M., Seattle policeman Todd Harris was on a routine patrol. He passed a car that was parked along the curb. The parking lights were on. The windows were steamed up, but it appeared there were two occupants.

There was no sign of misbehavior. Nevertheless, Harris noted the Washington license plate number as he drove by. As he continued on his patrol, he ran the number through his computer to make sure the car had not been reported stolen. Several blocks later the name of the registered owner came onto the screen. The car was legally registered and was not stolen, but he recognized the name from reading about the case in the newspapers.

Mary K. LeTourneau.

Officer Harris returned, and asked for ID. The two occupants were Mary and the boy.

The boy was taken home. Mary was taken to the station, then arrested for violation of parole. On Friday, February 6, 1998, the judge revoked her parole and sentenced her to serve the full eighty-nine months. Leaving aside the morality of meeting with the father of her child, here is the lesson I want you to draw from this story: *At the time Mary titled the car,* there was not—could not have function for this system is a matrix valued equation of the form

$$\left(\frac{d}{dt}\overleftrightarrow{1}-\overleftrightarrow{W}\right)\overleftrightarrow{G}(t-t')=\delta(t-t')\overleftrightarrow{1}.$$

?

a. Show that the equation of motion is solved by $\vec{\eta}(t)=\int_{-\infty}^{t}dt'\overleftrightarrow{G}(t-t')\vec{f}(t')$.

b. The Green's function depends on the boundary conditions. For the case where $\overleftrightarrow{G}(t-t')=0$ for all t less than t' show that $\overleftrightarrow{G}(t-t')=0$
$\overleftrightarrow{G}(t-t')=\exp\left(\overleftrightarrow{W}(t-t')\right)\theta(t-t')$

c. Writing $\overleftrightarrow{W}=\overleftrightarrow{S}^{-1}\overleftrightarrow{W}_d\overleftrightarrow{S}$ where $\overleftrightarrow{S}$ is the similarity matrix which diagonalizes $\overleftrightarrow{W}$ and $\overleftrightarrow{W}_d$ is the diagonal matrix of eigenvalues of $\overleftrightarrow{W}$ show that
$\overleftrightarrow{G}(t-t')=\overleftrightarrow{S}^{-1}\exp\left(\overleftrightarrow{W}_d(t-t')\right)\overleftrightarrow{S}\,\theta(t-t')$

been—the slightest indication of the troubles that lay ahead. But who among us can guarantee that quiet waters will never see a storm? [See my own experiences in Chapter 13.)

After the LeTourneau sentencing, I sent a notice to all my clients in the United States and Canada with this headline:

SEVEN YEARS IN JAIL FOR NOT
USING A LIMITED LIABILITY COMPANY!

You will often hear that a limited-liability company is something *"new,"* but this refers only to the United States. In Europe, LLCs have been used for more than a century and, as mentioned previously, I myself formed my first one in Spain in 1972. As I said in Chapter 10, "Think of a corporation as a brother whose twin fraternal sister is an LLC. Both persons are very similar, same family, same traits. It's just that under certain circumstances the anatomical differences must be considered." So then, let's discuss anatomical differences as applied strictly to limited-liability companies.

Think of an LLC primarily as a partnership, but without the liability and (now hear this!) without the necessity of actually having a partner. It also resembles a corporation but without many of the onerous bookkeeping details and annual meetings.

I've gone over the LLC statutes in virtually every state, and by no means are they all the same. California, Illinois, and New York have outrageously high fees and are not recommended except in the most dire circumstances. Neither are any of the well-known states used for incorporation—the LLC laws are entirely different. I have discovered several states that—although unsuitable for incorporating, are ideal for LLCs, because the *only* information required in the Articles of Organization is:

1. The name of the company,

2. The name and address of the resident agent (to be explained shortly), and

3. The duration of the LLC.

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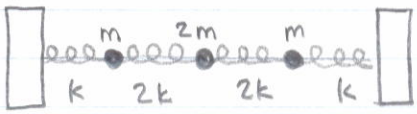
1. 

a) Normal modes? Associated frequencies?

$$L = \frac{1}{2}(m\dot{x}_1^2 + 2m\dot{x}_2^2 + m\dot{x}_3^2) - \frac{1}{2}(kx_1^2 + 2k(x_2-x_1)^2 + 2k(x_3-x_2)^2 + kx_3^2)$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$\overleftrightarrow{T} = \begin{pmatrix} m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & m \end{pmatrix} \qquad \overleftrightarrow{V} = \begin{pmatrix} 3k & -2k & 0 \\ -2k & 4k & -2k \\ 0 & -2k & 3k \end{pmatrix}$$

$$-\omega_n^2 \vec{X}_n = \overleftrightarrow{T}^{-1}\overleftrightarrow{V}\vec{X}_n$$

$$\overleftrightarrow{T}^{-1}\overleftrightarrow{V} = \omega_0^2 \begin{pmatrix} 3 & -2 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 3 \end{pmatrix} \quad , \quad \omega_0 = \sqrt{\frac{k}{m}}$$

See Mathematica

$$\omega_1 = \sqrt{3}\omega_0 \qquad \vec{X}_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\omega_2 = \sqrt{\frac{5-\sqrt{17}}{2}}\,\omega_0 \qquad \vec{X}_2 = \begin{pmatrix} 1 \\ \frac{1}{4}(1+\sqrt{17}) \\ 1 \end{pmatrix}$$

$$\omega_3 = \sqrt{\frac{5+\sqrt{17}}{2}}\,\omega_0 \qquad \vec{X}_3 = \begin{pmatrix} 1 \\ \frac{1}{4}(1-\sqrt{17}) \\ 1 \end{pmatrix}$$

b) Verify Orthogonality: $\vec{X}_m^T \overleftrightarrow{T} \vec{X}_n = 0$, $\omega_m \neq \omega_n$

$$\vec{X}_1^T \overleftrightarrow{T} \vec{X}_2 = (-1 \;\; 0 \;\; 1)\begin{pmatrix} m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & m \end{pmatrix}\begin{pmatrix} 1 \\ \frac{1}{4}(1+\sqrt{17}) \\ 1 \end{pmatrix} = 0$$

Similarly, using Mathematica

$$\vec{X}_1^T \overleftrightarrow{T}\vec{X}_3 = 0 \qquad \vec{X}_2^T \overleftrightarrow{T}\vec{X}_3 = 0$$

1c) at $t=0$, all particles at equilibrium position
1 & 3 at rest, 2 moving w/ velocity v

ie.

$$\vec{x}(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \dot{\vec{x}}(0) = \begin{pmatrix} 0 \\ v \\ 0 \end{pmatrix}$$

$\vec{x}(t) = ?$

$$\vec{x}(t) = \sum_n A_n e^{i\omega_n t}\, \vec{x}_n$$

$$\begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = A_1 \cancelto{1 \text{ at } t=0}{e^{i\sqrt{3}\omega_0 t}} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + A_2 \cancelto{1}{e^{i\sqrt{\frac{5-\sqrt{17}}{2}}\omega_0 t}} \begin{pmatrix} 1 \\ \frac{1}{4}(1+\sqrt{17}) \\ 1 \end{pmatrix} + A_3 \cancelto{1}{e^{i\sqrt{\frac{5+\sqrt{17}}{2}}\omega_0 t}} \begin{pmatrix} 1 \\ \frac{1}{4}(1-\sqrt{17}) \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} \dot{x}_1(0) \\ \dot{x}_2(0) \\ \dot{x}_3(0) \end{pmatrix} = \begin{pmatrix} 0 \\ v \\ 0 \end{pmatrix} = A_1\, i\sqrt{3}\omega_0 \cancelto{1}{e^{i\sqrt{3}\omega_0 t}} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + A_2\, i\sqrt{\frac{5-\sqrt{17}}{2}}\,\omega_0 \begin{pmatrix} 1 \\ \frac{1}{4}(1+\sqrt{17}) \\ 1 \end{pmatrix}$$

$$+ A_3\, i\sqrt{\frac{5+\sqrt{17}}{2}}\,\omega_0 \begin{pmatrix} 1 \\ \frac{1}{4}(1-\sqrt{17}) \\ 1 \end{pmatrix}$$

$$\begin{cases} -A_1 + A_2 + A_3 = 0 \\ A_2\left(\frac{1}{4}(1+\sqrt{17})\right) + A_3\left(\frac{1}{4}(1-\sqrt{17})\right) = 0 \\ A_1 + A_2 + A_3 = 0 \end{cases}$$

$$(1+\sqrt{17})A_2 = -(1-\sqrt{17})A_3$$

but

$$2(A_2 + A_3) = 0$$

so $Re(A_1) = Re(A_2) = Re(A_3) = 0$

1c) cont

$$0 = -A_1 i\sqrt{3}\omega_0 + A_2 i\sqrt{\frac{5-\sqrt{17}}{2}}\omega_0 + A_3 i\sqrt{\frac{5+\sqrt{17}}{2}}\omega_0$$

$$V = A_2 i\sqrt{\frac{5-\sqrt{17}}{2}}\omega_0\left(\frac{1}{4}(1+\sqrt{17})\right) + A_3 i\sqrt{\frac{5+\sqrt{17}}{2}}\omega_0 \frac{1}{4}(1-\sqrt{17})$$

$$0 = A_1 i\sqrt{3}\omega_0 + A_2 i\sqrt{\frac{5-\sqrt{17}}{2}}\omega_0 + A_3 i\sqrt{\frac{5+\sqrt{17}}{2}}\omega_0$$

$+$

$$0 = \cancel{2}A_2 i\sqrt{\frac{5-\sqrt{17}}{2}}\omega_0 + \cancel{2}A_3 i\sqrt{\frac{5+\sqrt{17}}{2}}\omega_0$$

$$\sqrt{\frac{5-\sqrt{17}}{2}}A_2 = -A_3\sqrt{\frac{5+\sqrt{17}}{2}}$$

$$A_2 = -\sqrt{\frac{5+\sqrt{17}}{2}}\sqrt{\frac{2}{5-\sqrt{17}}}A_3 = -\sqrt{\frac{(5+\sqrt{17})^2}{8}}A_3 = \frac{-(5+\sqrt{17})}{\sqrt{8}}A_3$$

$$V = A_2\sqrt{\frac{5-\sqrt{17}}{2}}\, i\omega_0 \frac{1}{4}(1+\sqrt{17}) + A_3 i\sqrt{\frac{5+\sqrt{17}}{2}}\omega_0\frac{1}{4}(1-\sqrt{17})$$

$$= \frac{i\omega_0}{4}\left[-A_3\sqrt{\frac{5+\sqrt{17}}{2}}(1+\sqrt{17}) + A_3 i\sqrt{\frac{5+\sqrt{17}}{2}}(1-\sqrt{17})\right]$$

$$= \frac{A_3 i\omega_0}{4}\sqrt{\frac{5+\sqrt{17}}{2}}\left[\cancel{-1}-\sqrt{17}\cancel{+1}-\sqrt{17}\right]$$

$$V = \frac{-2\sqrt{17}\, i\omega_0}{4}\sqrt{\frac{5+\sqrt{17}}{2}}A_3$$

$$A_3 = \frac{-4}{2\sqrt{17}\, i\omega_0}\sqrt{\frac{2}{5+\sqrt{17}}}V = \frac{2i}{\sqrt{17}\omega_0}\sqrt{\frac{2}{5+\sqrt{17}}}V$$

$$A_2 = -\left(\frac{5+\sqrt{17}}{\sqrt{8}}\right)A_3 = \frac{-2i}{\omega_0}\frac{5+\sqrt{17}}{\sqrt{17}\sqrt{8}}\sqrt{\frac{2}{5+\sqrt{17}}}V = \frac{-i}{\omega_0}\sqrt{\frac{5+\sqrt{17}}{17}}V$$

$A_1 = 0$

1c cont

$$\mathrm{Re}\left(A_2\, e^{i\sqrt{\frac{5-\sqrt{17}}{2}}\omega_0 t}\right) = A_2\left(i \sin\sqrt{\frac{5-\sqrt{17}}{2}}\,\omega_0 t\right)$$

$$= \frac{-i}{\omega_0}\sqrt{\frac{5+\sqrt{17}}{17}}\, v \left(i \sin\sqrt{\frac{5-\sqrt{17}}{2}}\,\omega_0 t\right)$$

$$= \frac{v}{\omega_0}\sqrt{\frac{5+\sqrt{17}}{17}} \sin\sqrt{\frac{5-\sqrt{17}}{2}}\,\omega_0 t$$

$$\mathrm{Re}\left(A_3\, e^{i\sqrt{\frac{5+\sqrt{17}}{2}}\omega_0 t}\right) = \frac{2i}{\sqrt{17}\,\omega_0}\sqrt{\frac{2}{5+\sqrt{17}}}\, v\, i \sin\sqrt{\frac{5+\sqrt{17}}{2}}\,\omega_0 t$$

$$= -\frac{2v}{\sqrt{17}\,\omega_0}\sqrt{\frac{2}{5+\sqrt{17}}} \sin\sqrt{\frac{5+\sqrt{17}}{2}}\ \omega_0 t$$

$\vec{x}(t)$:

$$\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = \frac{v}{\omega_0}\sqrt{\frac{5+\sqrt{17}}{17}} \sin\left(\sqrt{\frac{5-\sqrt{17}}{2}}\,\omega_0 t\right)\begin{pmatrix} 1 \\ \frac{1}{4}(1+\sqrt{17}) \\ 1 \end{pmatrix}$$

$$- \frac{2v}{\sqrt{17}\,\omega_0}\sqrt{\frac{2}{5+\sqrt{17}}} \sin\left(\frac{5+\sqrt{17}}{2}\,\omega_0 t\right)\begin{pmatrix} 1 \\ \frac{1}{4}(1-\sqrt{17}) \\ 1 \end{pmatrix}$$

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2. system from problem 1

$$\overleftrightarrow{T}\,\ddot{\vec{x}} + \overleftrightarrow{V} x = \vec{f}(t)$$

w/ $\vec{f}(t) = \begin{pmatrix} 0 \\ f_0 \sinh(t/T)\,\theta(t) \\ 0 \end{pmatrix}$

$$\overleftrightarrow{T} = \begin{pmatrix} m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & m \end{pmatrix} \qquad \overleftrightarrow{V} = \begin{pmatrix} 3k & -2k & 0 \\ -2k & 4k & -2k \\ 0 & -2k & 3k \end{pmatrix}$$

$$\ddot{\vec{x}} = \overleftrightarrow{T}^{-1}\vec{f} - \overleftrightarrow{T}^{-1}\overleftrightarrow{V} x$$

or $\left(-\frac{\partial^2}{\partial t^2} + \overleftrightarrow{T}^{-1}\overleftrightarrow{V}\right)\vec{x}_n = \overleftrightarrow{T}^{-1}\vec{f}$

let $\overleftrightarrow{T}^{-1}\vec{f}(t) = \sum_n \phi_n(t)\,\vec{X}_n$

where $\vec{x}(t) = \sum_n c_n(t)\,\vec{X}_n$ $\Big\}$ $\vec{X}_n$ from problem 1

orthogonality: $x_m^T\, T\, x_n = 0$, $m \neq n$

$$\vec{X}_m \cdot \vec{f}(t) = \sum_n \phi_n\, \vec{X}_m^T\, \overleftrightarrow{T}\, \vec{X}_n = \phi_m\, \vec{X}_m^T\, \overleftrightarrow{T}\, \vec{X}_m$$

$$\phi_m = \frac{\vec{X}_m \cdot \vec{f}(t)}{\vec{X}_m^T\, \overleftrightarrow{T}\, \vec{X}_m}$$

$$\phi_1 = \frac{\vec{X}_1 \cdot \vec{f}(t)}{\vec{X}_1^T\, \overleftrightarrow{T}\, \vec{X}_1} = 0 \qquad \phi_2 = \frac{\vec{X}_2 \cdot \vec{f}}{\vec{X}_2^T\, \overleftrightarrow{T}\, \vec{X}_2} = \frac{f_0}{\sqrt{17}\, m}\sinh\left(\frac{t}{T}\right)\theta(t)$$

2 cont

$$\phi_3 = \frac{-f_0}{\sqrt{17}m} \sinh\left(\frac{t}{T}\right) \Theta(t)$$

from diff eqn:

$$\sum_n \left(-\frac{\partial}{\partial t^2}\mathbb{1} + \overleftrightarrow{T}^{-1}\overleftrightarrow{V}\right) c_n(t)\, \vec{x}_n = \sum_n \phi_n(t)\, \vec{x}_n$$

$$\downarrow$$

$$\left(\frac{-\partial^2}{\partial t^2} + \omega_n^2\right) c_n(t) = \phi_n(t) \quad \Big\}\ \text{familiar eqn}$$

$$\downarrow$$

$$c_n(t) = \int_{-\infty}^{\infty} dt'\, G_n(t-t')\, \phi_n(t')$$

$$G_n(t-t') = \frac{\sin(\omega_n(t-t'))}{\omega_n} \Theta(t-t')$$

from problem 1

$$\omega_1 = \sqrt{3}\,\omega_0 \qquad \omega_2 = \sqrt{\frac{5-\sqrt{17}}{2}}\,\omega_0 \qquad \omega_2 = \sqrt{\frac{5+\sqrt{17}}{2}}\,\omega_0$$

$$G_1 = \frac{\sin(\sqrt{3}\,\omega_0(t-t'))}{\sqrt{3}\,\omega_0} \Theta(t-t')$$

$$C_1(t) = \int_{-\infty}^{\infty} dt' \frac{\sin(\sqrt{3}\,\omega_0(t-t'))}{\sqrt{3}\,\omega_0} \Theta(t-t')\, \underbrace{\phi_1}_{0}$$

$$C_1(t) = 0$$

$$C_2(t) = \int_{-\infty}^{\infty} dt' \frac{\sin(\omega_2(t-t'))}{\omega_2} \Theta(t-t')\, \frac{f_0}{\sqrt{17}m} \sinh\left(\frac{t'}{T}\right) \Theta(t')$$

2 cont

$$C_2(t) = \frac{f_0}{\omega_2 m\sqrt{17}} \int_{-\infty}^{\infty} \sin[\omega_2(t-t')] \sinh\left(\frac{t'}{T}\right) \underbrace{\Theta(t-t')\Theta(t')}_{\text{changes limits of integration}} dt'$$

$$= \frac{f_0}{\omega_2 m\sqrt{17}} \int_0^t \sin(\omega_2(t-t')) \sinh\left(\frac{t'}{T}\right) dt'$$

$$C_2 = \frac{f_0}{\omega_2 m\sqrt{17}} \frac{T}{1+\omega_2^2 T^2} \left(\omega_2 T \sinh\left(\frac{t}{T}\right) - \sin(\omega_2 t)\right)$$

$$C_3 = \int_{-\infty}^{\infty} dt' \frac{\sin(\omega_3(t-t'))}{\omega_3} \Theta(t-t') \frac{-f_0}{\sqrt{17} m} \sinh\left(\frac{t'}{T}\right) \Theta(t')$$

$$= \frac{-f_0}{\sqrt{17} m \omega_3} \int_{-\infty}^{\infty} \sin[\omega_3(t-t')] \sinh\left(\frac{t'}{T}\right) \Theta(t-t') \Theta(t') dt'$$

$$= \frac{-f_0}{\sqrt{17} m \omega_3} \int_0^t \sin[\omega_3(t-t')) \sinh\left(\frac{t'}{T}\right) dt'$$

$$= \frac{-f_0}{\omega_2 m\sqrt{17}} \frac{T}{1+\omega_3^2 T^2} \left(\omega_3 T \sinh\left(\frac{t}{T}\right) - \sin(\omega_3 t)\right)$$

$$\vec{X}(t) = \cancel{C_1} \overset{\to 0}{\vec{X}_1} + C_2 \vec{X}_2 + C_3 \vec{X}_3 =$$

$$\vec{X}(t) = \frac{f_0 T}{\sqrt{17} m}\left[\frac{\omega_2 T \sinh\left(\frac{t}{T}\right) - \sin(\omega_2 t)}{\omega_2(1+\omega_2^2 T^2)} \begin{pmatrix} 1 \\ \frac{1}{4}(1+\sqrt{17}) \\ 1 \end{pmatrix} - \frac{\omega_3 T \sinh\left(\frac{t}{T}\right) - \sin(\omega_3 t)}{\omega_3(1+\omega_3^2 T^2)} \begin{pmatrix} 1 \\ \frac{1}{4}(1-\sqrt{17}) \\ 1 \end{pmatrix} \right]$$

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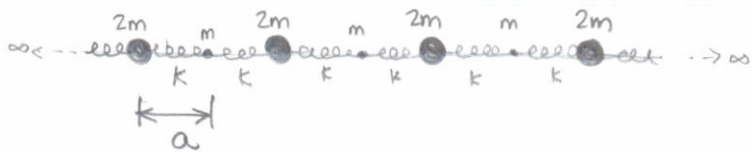
3. 

dispersion relation?

$$T = m\begin{pmatrix} \ddots & & & & \\ & m & & 0 & \\ & & 2m & & \\ & 0 & & m & \\ & & & & 2m \ddots \end{pmatrix} \qquad V = \begin{pmatrix} \ddots & & & & 0 \\ & 2 & -1 & 0 & 0 \\ & -1 & 2 & -1 & 0 \\ & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \ddots \end{pmatrix}$$

$$T_{ij} = \begin{cases} m\,\delta_{ij} & i \text{ even} \\ 2m\,\delta_{ij} & i \text{ odd} \end{cases} \qquad V_{ij} = k\left[2\delta_{ij} - \delta_{i+1\,j} - \delta_{i-1\,j}\right]$$

normal modes

$$\omega_n^2 \vec{x}_n = \overleftrightarrow{T}^{-1} \overleftrightarrow{V} x_n$$

Symmetry: translation by 2a

$$[\overleftrightarrow{S}, \overleftrightarrow{\Phi}] = 0 \qquad \Phi \vec{x}_n = \omega_n^2 \vec{x}_n$$
$$S \vec{x}_n = s_n \vec{x}_n$$

$$\vec{X} = \begin{pmatrix} \vdots \\ x_{n_i} \\ x_{n_{i+1}} \\ \vdots \end{pmatrix} \qquad X_{n_{i+2}} = S_n X_{n_i}$$

$$\text{so, } x_{n_j} = (s_n)^j \, |x_{j-2}|$$

need $|s_n| = 1$ so x_{n_i} doesn't blow up as $j \to \infty$

let $S_n = e^{i\alpha_n}$ $\qquad \alpha_n = k_n a$

$$X_j = (e^{ika})^j \, |X_{j-2}|$$

3 cont need to pick $|x_0\rangle$ & $|x_1\rangle$, then have rest from the relationship $x_{j+2} = e^{ika}|x_{j-2}|$

$$x_0 = \alpha \qquad x_1 = \beta e^{ika}$$

so:

$$x_2 = \alpha e^{2ika} \qquad x_3 = \beta e^{3ika} \qquad \ldots \text{etc}$$

$$\overleftrightarrow{T}^{-1}\overleftrightarrow{V} = \begin{pmatrix} 1/m & & & 0 \\ & 1/2m & & \\ & & 1/m & \\ 0 & & & 1/2m \end{pmatrix}\begin{pmatrix} 2 & -1 & & 0 \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ 0 & & & \ddots \end{pmatrix}$$

$$= \omega_0^2 \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1/2 & 1 & -1/2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1/2 & 1 \end{pmatrix}$$

$$\omega_0^2 \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1/2 & 1 & -1/2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1/2 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} \beta e^{ika} \\ \alpha \\ \beta e^{-ika} \\ \alpha e^{-2ika} \end{pmatrix}}_{\vec{x}_k} = \begin{pmatrix} -\alpha e^{2ika} + 2\beta e^{ika} - \alpha \\ -\frac{1}{2}\beta e^{ika} + \alpha - \frac{1}{2}\beta e^{-ika} \\ -\alpha + 2\beta e^{-ika} - \alpha e^{-2ika} \\ -\frac{1}{2}\beta e^{-ika} + \alpha e^{-2ika} - \frac{1}{2}\beta e^{-3ika} \end{pmatrix} \omega_0^2$$

$$= \omega_0^2 \begin{pmatrix} \left(-\frac{\alpha}{\beta}e^{ika} + 2 - \frac{\alpha}{2}e^{-ika}\right)\beta e^{ika} \\ \left(-\frac{\beta}{2\alpha}e^{ika} + 1 - \frac{\beta}{2\alpha}e^{-ika}\right)\alpha \\ \left(-\frac{\alpha}{\beta}e^{ika} + 2 - \frac{\alpha}{\beta}e^{-ika}\right)\beta e^{-ika} \\ \left(-\frac{\beta}{2\alpha}e^{ika} + 1 - \frac{\beta}{2\alpha}e^{-ika}\right)\alpha e^{-ika} \end{pmatrix}$$

to satisfy $\omega_k^2 \vec{x}_k = \overleftrightarrow{T}^{-1}\overleftrightarrow{V}\vec{x}_k$ we need

$$\omega_k^2 = \omega_0^2\left(-\frac{\alpha}{\beta}e^{ika} + 2 - \frac{\alpha}{\beta}e^{-ika}\right)$$
$$= \omega_0^2\left(-\frac{\beta}{2\alpha}e^{ika} + 1 - \frac{\beta}{2\alpha}e^{-ika}\right)$$

3 cont

satisfied if:

$$-\frac{\alpha}{\beta}\left(e^{ika}+e^{-ika}\right)+2 = \frac{-\beta}{2\alpha}\left(e^{ika}+e^{-ika}\right)+1$$

$$\cdot -\alpha\beta \quad \left(-2\frac{\alpha}{\beta}\cos ka + 1 = -\frac{\beta}{\alpha}\cos ka\right)$$

$$\longrightarrow \quad 2\alpha^2 \cos ka - \alpha\beta - \beta^2 \cos ka = 0$$

$$\alpha = \frac{\beta \pm \sqrt{\beta^2 - 4(2\cos ka)(-\beta^2\cos ka)}}{2(2\cos ka)}$$

$$\alpha = \frac{\beta\left(1 \pm \sqrt{1+8\cos^2 ka}\right)}{2(2\cos ka)}$$

$$\frac{\alpha}{\beta} = \frac{1 \pm \sqrt{1+8\cos^2 ka}}{2(2\cos ka)}$$

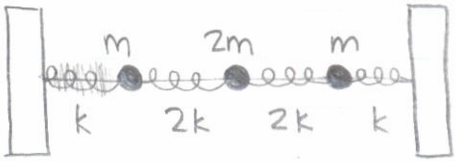
$$\omega_k^2 = \omega_0^2\left(2 - 2\frac{\alpha}{\beta}\cos ka\right)$$

$$\omega_k^2 = \omega_0^2\left(2 - 2\cos ka \cdot \frac{1 \pm \sqrt{1+8\cos^2}}{2(2\cos ka)}\right) = \omega_0^2\left(2 - \left(\frac{1}{2} \pm \frac{\sqrt{1+8\cos^2 ka}}{2}\right)\right)$$

$$\omega_k^2 = \omega_0^2\left(\frac{3}{2} \pm \sqrt{1+8\cos^2 ka}\,/2\right)$$

$$\boxed{\omega_k = \omega_0\left(\frac{3}{2} \pm \frac{\sqrt{1+8\cos^2 ka}}{2}\right)^{1/2}}$$

10

4. 

Shock absorber: $F = -m\omega_0 \dot{x}_1$, $\omega_0^2 = \frac{k}{m}$

$$\vec{\eta} = \begin{pmatrix} \vec{q} \\ \vec{p} \end{pmatrix} \qquad \overleftrightarrow{W} = \begin{pmatrix} 0 & \overleftrightarrow{T}^{-1} \\ -\overleftrightarrow{V} & -\overleftrightarrow{F}\,\overleftrightarrow{T}^{-1} \end{pmatrix} \qquad \dot{\vec{\eta}} = \overleftrightarrow{W}\vec{\eta}$$

$$\vec{\eta}(t) = \sum_n \vec{\eta}_n e^{i\omega_n t}$$, Find $\vec{\eta}_n$

Same system as problem 1 + damping
↳ T and V should be the same

$$\overleftrightarrow{T} = m\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \overleftrightarrow{V} = k\begin{pmatrix} 3 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 3 \end{pmatrix}$$

$$\overleftrightarrow{T}\ddot{\vec{x}} = -\overleftrightarrow{V}\vec{x} - \overleftrightarrow{F}\dot{\vec{x}}$$

$$\overleftrightarrow{F} = \begin{pmatrix} m\omega_0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$-\overleftrightarrow{T}^{-1}\overleftrightarrow{V}\vec{q} - \overleftrightarrow{T}^{-1}\overleftrightarrow{F}\dot{\vec{q}} = \ddot{\vec{q}}$$

$$m\ddot{x} = -Vx - F\dot{x}$$

$$\overleftrightarrow{T}^{-1} = \frac{1}{m}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$-\overleftrightarrow{F}\,\overleftrightarrow{T}^{-1} = -\frac{1}{m}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} m\omega_0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{-1}{m}\begin{pmatrix} m\omega_0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -\omega_0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

4cont $\overleftrightarrow{W} = \begin{pmatrix} 0 & 0 & 0 & 1/m & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2m & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/m \\ -3k & 2k & 0 & -\omega_0 & 0 & 0 \\ 2k & -4k & 2k & 0 & 0 & 0 \\ 0 & 2k & -3k & 0 & 0 & 0 \end{pmatrix}$

want solutions of the form $i\omega_n \vec{\eta}_n = \overleftrightarrow{W} \vec{\eta}_n$

use Mathematica → messy b/c of constants

define $a \equiv \sqrt{mk}$

then $1/m = \sqrt{k/m} \frac{1}{\sqrt{mk}} = \omega_0 / a$

$k = \sqrt{k/m} \sqrt{mk} = \omega_0 a$

so ω_0 factors out of $\overleftrightarrow{W}$

$$\overleftrightarrow{W} = \omega_0 \begin{pmatrix} 0 & 0 & 0 & 1/a & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2a & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/a \\ -3a & 2a & 0 & -1 & 0 & 0 \\ 2a & -4a & 2a & 0 & 0 & 0 \\ 0 & 2a & -3a & 0 & 0 & 0 \end{pmatrix} \quad = \overleftrightarrow{W}_2 \text{ on Mathematica}$$

Mathematica can numerically solve this eigensystem:

$\omega_1 = -2.04 + 0.078i$

$$\eta_1 = \begin{bmatrix} (0.256 - 0.106i)/a \\ -(0.067 - 0.286i)/a \\ -(0.018 + 0.489i)/a \\ 0.195 + 0.532i \\ -1.156 - 0.322i \\ 1 \end{bmatrix}$$

$\omega_2 = +2.04 + 0.078i$

$$\eta_2 = \begin{bmatrix} (0.256 + 0.106i)/a \\ -(0.067 + 0.286i)/a \\ -(0.018 - 0.489i)/a \\ 0.195 - 0.532i \\ -1.156 + 0.322i \\ 1 \end{bmatrix}$$

4 cont

$\omega_3 = -1.73 + 0.324i$

$$\eta_3 = \begin{bmatrix} (0.425 + 0.887i)/a \\ -(0.318 - 0.030i)/a \\ -(0.104 + 0.557i)/a \\ -1.675 + 0.449i \\ 0.101 - 1.123i \\ 1 \end{bmatrix}$$

$\omega_4 = 1.73 + 0.324i$

$$\eta_4 = \begin{bmatrix} (0.425 - 0.887i)/a \\ -(0.318 + 0.030i)/a \\ -(0.104 - 0.557i)/a \\ -1.675 - 0.449i \\ 0.101 + 1.123i \\ 1 \end{bmatrix}$$

$\omega_5 = -0.673 + 0.097i$

$$\eta_5 = \begin{bmatrix} -(0.598 + 1.369i)/a \\ -(0.364 + 1.845i)/a \\ -(0.210 + 1.45i)/a \\ 0.980 - 0.269i \\ 2.556 - 0.13i \\ 1 \end{bmatrix}$$

$\omega_6 = 0.673 + 0.097i$

$$\eta_6 = \begin{bmatrix} -(0.598 - 1.369i)/a \\ -(0.364 - 1.845i)/a \\ -(0.210 - 1.45i)/a \\ 0.980 + 0.269i \\ 2.556 + 0.13i \\ 1 \end{bmatrix}$$

Here we have 6 eigenvectors, but the masses should only have 3 modes. Notice that the eigenvectors came in pairs however. Each pair has a generalized relationship:

$$\eta_j e^{i\omega_j t} + \eta_{j+1} e^{i\omega_{j+1} t} = (\vec{\alpha} + i\vec{\beta}) e^{(-c-id)t} + (\vec{\alpha} - i\vec{\beta}) e^{(-c+id)t}$$

this can be simplified to:

$$e^{-ct}\left[(\vec{\alpha} + i\vec{\beta})e^{-idt} + (\alpha - i\vec{\beta})e^{idt}\right]$$

$$= e^{-ct}\left[\vec{\alpha}(e^{-idt} - e^{idt}) + i\vec{\beta}(e^{-idt} - e^{idt})\right]$$

$$= e^{-ct}\left[2\vec{\alpha}\cos dt + 2\vec{\beta}\sin dt\right]$$

The six eigenvectors form 3 pairs which give three oscillating but decaying modes, as to be expected.

5. System from problem 4

middle particle charged (q)

magnetic field, $E = E_0 \cos(2\omega_0 t)$

direction of $E \rightarrow$ w/ respect to 

driving force on middle particle:

$$F = qE$$

$$\dot{\vec{\eta}} = \overleftrightarrow{W}\vec{\eta} + \vec{f}(t)$$

$$\vec{f}(t) = \begin{pmatrix} \vec{0} \\ \vec{F} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ qE \\ 0 \end{pmatrix} \qquad qE = qE_0 \cos(2\omega_0 t)$$

let $\vec{\eta}(t)$ be of the form $\vec{\eta}_0 e^{2i\omega_0 t}$

at steady state

$$\dot{\vec{\eta}} = \frac{d}{dt}\left(\vec{\eta}_0 e^{2i\omega_0 t}\right) = 2i\omega_0 \vec{\eta}_0 e^{2i\omega_0 t} = 2i\omega_0 \vec{\eta}$$

$$\left(\frac{d}{dt}\mathbb{1} - \overleftrightarrow{W}\right)\vec{\eta}(t) = \vec{f}(t)$$

$$\left(2i\omega_0 - \overleftrightarrow{W}\right)\vec{\eta}(t) = \vec{f}(t)$$

want to find $\vec{\eta}_0$, at $t=0$ $\vec{\eta}(0) = \eta_0$ $\vec{f}(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ qE_0 \\ 0 \end{pmatrix}$

$$(\underbrace{2i\omega_0 - \overleftrightarrow{W}}_{B})\,\vec{\eta}_0 = \vec{f}(0)$$

$$\vec{\eta}_0 = \left(2i\omega_0 - \overleftrightarrow{W}\right)^{-1}\vec{f}(0)$$

5 cont $\quad \vec{\eta}_0 = \begin{pmatrix} \vec{q}_0 \\ \vec{p}_0 \end{pmatrix}$

from Mathematica, $\vec{q}_0 = \frac{1}{\omega_0} \begin{pmatrix} -E_0 q/2a \\ \frac{1}{a}\left(\frac{1}{4} - \frac{i}{2}\right) E_0 q \\ -\left(\frac{1}{2} - i\right) E_0 q \frac{1}{a} \end{pmatrix}$

$$\vec{q}_0 = \frac{E_0 q}{\omega_0 a} \begin{pmatrix} -1/2 \\ \left(\frac{1}{4} - \frac{i}{2}\right) \\ \left(i - 1/2\right) \end{pmatrix}$$

$$\vec{q}(t) = Re\left(\vec{q}_0 e^{2i\omega_0 t}\right)$$

$$= Re\left[\begin{pmatrix} -\frac{1}{2} \\ \frac{1}{4} - \frac{i}{2} \\ i - \frac{1}{2} \end{pmatrix} \left(\cos 2\omega_0 t + i \sin 2\omega_0 t\right)\right]$$

$$\boxed{\vec{q}(t) = \begin{pmatrix} -\frac{1}{2}\cos 2\omega_0 t \\ \frac{1}{4}\cos 2\omega_0 t + \frac{1}{2}\sin 2\omega_0 t \\ -\frac{1}{2}\cos 2\omega_0 t - \sin 2\omega_0 t \end{pmatrix}}$$

6. $\vec{\eta} = \begin{pmatrix} \vec{q} \\ \vec{p} \end{pmatrix}$ $\quad \overleftrightarrow{W} = \begin{pmatrix} 0 & \overleftrightarrow{T}^{-1} \\ -\overleftrightarrow{V} & -\overleftrightarrow{F}\overleftrightarrow{T}^{-1} \end{pmatrix}$

undriven: $\dot{\vec{\eta}} = \overleftrightarrow{W}\vec{\eta}$

$\vec{\eta}(t) = \vec{\eta}_n e^{i\omega_n t} \rightarrow i\omega_n \vec{\eta}_n = \overleftrightarrow{W}\vec{\eta}_n$

driven:
$\dot{\vec{\eta}} = \overleftrightarrow{W}\vec{\eta} + \vec{f}(t)$, $\vec{f}(t) = \begin{pmatrix} 0 \\ \vec{F}(t) \end{pmatrix}$

Green's Function:
$$\left(\frac{d}{dt}\mathbb{1} - \overleftrightarrow{W}\right)\overleftrightarrow{G}(t-t') = \delta(t-t')\mathbb{1}$$

a) Show that $\vec{\eta}(t) = \int_{-\infty}^{t} dt' \overleftrightarrow{G}(t-t')\vec{f}(t')$

$$\dot{\vec{\eta}} - \overleftrightarrow{W}\vec{\eta} = \vec{f}(t)$$
$$\downarrow$$
$$\left(\frac{d}{dt}\mathbb{1} - \overleftrightarrow{W}\right)\vec{\eta} = \vec{f}(t)$$

assume $\vec{\eta}(t)$ above is correct and show that it works

$$\left(\frac{d}{dt}\mathbb{1} - \overleftrightarrow{W}\right)\int_{-\infty}^{t} dt' \overleftrightarrow{G}(t-t')\vec{f}(t')$$

$$= \int_{-\infty}^{t} dt' \left(\frac{d}{dt}\mathbb{1} - \overleftrightarrow{W}\right)\overleftrightarrow{G}(t-t')\vec{f}(t')$$

$$= \int_{-\infty}^{t} dt' \delta(t-t')\vec{f}(t') = \vec{f}(t) \quad \checkmark$$

6.b) if $\overleftrightarrow{G}(t-t')=0$ for $t<t'$ show that

$$\overleftrightarrow{G}(t-t') = e^{\overleftrightarrow{W}(t-t')}\,\theta(t-t')$$

Green's Function:

$$\left(\frac{d}{dt}\mathbb{1} - \overleftrightarrow{W}\right)\overleftrightarrow{G} = \delta(t-t')\,\mathbb{1}$$

$$\left(\frac{d}{dt}\mathbb{1} - \overleftrightarrow{W}\right) e^{\overleftrightarrow{W}(t-t')}\,\theta(t-t')$$

$$= \overleftrightarrow{W} e^{\overleftrightarrow{W}(t-t')}\,\theta(t-t') + e^{\overleftrightarrow{W}(t-t')}\frac{d}{dt}\theta(t-t')$$

$$- \overleftrightarrow{W} e^{\overleftrightarrow{W}(t-t')}\,\theta(t-t')$$

$$= e^{\overleftrightarrow{W}(t-t')}\,\delta(t-t') \stackrel{?}{=} \delta(t-t')\,\mathbb{1}$$

for $t \neq t'$ we have $0 = 0$ ✓

for $t = t'$?

$$\int_{-\infty}^{\infty} e^{\overleftrightarrow{W}(t-t')}\,\delta(t-t') = \int_{-\infty}^{\infty}\delta(t-t')\,\mathbb{1}$$

$$e^{\overleftrightarrow{W}(t'-t')} = \mathbb{1}$$

$$e^{\overleftrightarrow{0}} = \mathbb{1} \quad ✓$$

(6c) Write $\overleftrightarrow{W} = \overleftrightarrow{S}^{-1} \underbrace{\overleftrightarrow{W}_d}_{\text{diagonal matrix of eigenvalues}} \underbrace{\overleftrightarrow{S}}_{\text{similarity matrix}}$

Show that $\overleftrightarrow{G}(t-t') = \overleftrightarrow{S}^{-1} e^{\overleftrightarrow{W}_d (t-t')} \overleftrightarrow{S}\, \theta(t-t')$

$$e^{\overleftrightarrow{W}} = 1 + \overleftrightarrow{W} + \frac{1}{2!}\overleftrightarrow{W}^2 + \frac{1}{3!}\overleftrightarrow{W}^3 + \cdots$$

$$\overleftrightarrow{W}^2 = \overleftrightarrow{S}^{-1}\overleftrightarrow{W}_d \overleftrightarrow{S}\,\overleftrightarrow{S}^{-1}\overleftrightarrow{W}_d\overleftrightarrow{S} = \overleftrightarrow{S}^{-1}\overleftrightarrow{W}_d^2\overleftrightarrow{S}$$

similarly, $\overleftrightarrow{W}^n = \overleftrightarrow{S}^{-1}\overleftrightarrow{W}_d^n\overleftrightarrow{S}$

$$e^{\overleftrightarrow{W}(t-t')} = \sum_{n=0}^{\infty} \frac{[\overleftrightarrow{W}(t-t')]^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{\overleftrightarrow{W}^n (t-t')^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{\overleftrightarrow{S}^{-1}\overleftrightarrow{W}_d^n\overleftrightarrow{S}(t-t')^n}{n!}$$

$$= \overleftrightarrow{S}^{-1}\sum_{n=0}^{\infty}\frac{\overleftrightarrow{W}_d^n (t-t')^n}{n!}\overleftrightarrow{S}$$

$$= \overleftrightarrow{S}^{-1} e^{\overleftrightarrow{W}_d(t-t')}\overleftrightarrow{S}$$

$$G(t-t') = e^{\overleftrightarrow{W}(t-t')}\theta(t-t')$$

$$= \overleftrightarrow{S}^{-1} e^{\overleftrightarrow{W}_d(t-t')}\overleftrightarrow{S}\,\theta(t-t') \quad \checkmark$$