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# Physics 601 Homework 5---Due Friday October 8

1. A standard result in undergraduate relativity is the velocity addition formula: $v = \frac{v_1 + v_2}{1 + v_1 v_2}$. This typically obtained by taking the product of two Lorentz transformations and the doing some algebra. A more straightforward way to obtain this use the four velocity: $u^\mu = \begin{pmatrix} \gamma \\ \gamma v_x \\ \gamma v_y \\ \gamma v_z \end{pmatrix}$ $\gamma = \frac{1}{\sqrt{1 - \vec{v}^2}}$. Suppose I start with a particle moving with velocity $\vec{v}_1$ with an associated four-velocity $u_1^\mu = \begin{pmatrix} \gamma_1 \\ \gamma_1 v_{1x} \\ \gamma_1 v_{1y} \\ \gamma_1 v_{1z} \end{pmatrix}$.

   Suppose one boosts to a new frame by running to the left (-*x* direction) with a velocity which corresponds to a Lorentz transformation $\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma_2 & v_2\gamma_2 & 0 & 0 \\ v_2\gamma_2 & \gamma_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

   The four-velocity in the new frame is $u^\nu = \Lambda^\mu{}_\nu u_1^\nu$.

   a. From the transformed four velocity find $v_x, v_y, v_z$ in the new frame.

   b. For the case where $\vec{v}_1$ is entirely along the x direction, show that $v = \frac{v_1 + v_2}{1 + v_1 v_2}$.

2. Consider relativistic transformations restricted to one spatial direction. In that case, the velocity can be specified by a single number from -1 to 1 (in units with *c*=1). It is convenient to introduce the "rapidity" $\eta$ with the property that $v = \tanh(\eta)$. Note that while *v* is restricted from -1 to 1, $\eta$ goes from -∞ to ∞.

   a. Show that the 4-velocity is given by $u^\mu = \begin{pmatrix} \cosh(\eta) \\ \sinh(\eta) \\ 0 \\ 0 \end{pmatrix}$.

   b. Show that while velocities do not add linearly in relativity rapidities do. That is one obtains the relativistic velocity addition formula by taking $\eta = \eta_1 + \eta_2$. In a certain sense this turns out to be just the hyperbolic trig version of the angle addition formula for two-dimensional rotations.

3. Relativistic Kinematics: Consider the elastic scattering of two particles with masses $m_1$ and $m_2$. Suppose initially particle 1 is at rest and and particle 2 approaches it

1. $u^\mu = \begin{pmatrix} \gamma \\ \gamma v_x \\ \gamma v_y \\ \gamma v_z \end{pmatrix}$ , $\gamma = \frac{1}{\sqrt{1-\vec{v}}}$

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start with a particle moving w/ $\vec{v}_1$ w/ $u_1^\mu = \begin{pmatrix} \gamma_1 \\ \gamma_1 v_{1x} \\ \gamma_1 v_{1y} \\ \gamma_1 v_{1z} \end{pmatrix}$

boost to new frame, $\Lambda^\mu_\nu = \begin{bmatrix} \gamma_2 & v_2\gamma_2 & 0 & 0 \\ v_2\gamma_2 & \gamma_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(by moving to left)

four velocity in new frame $= u^\nu = \Lambda^u_\nu u_1^\nu$

a) From $u^\nu$, find $v_x, v_y, v_z$ in new frame

$$u^\nu = \begin{bmatrix} \gamma_2 & v_2\gamma_2 & 0 & 0 \\ v_2\gamma_2 & \gamma_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_1 v_{1x} \\ \gamma_1 v_y \\ \gamma_1 v_{1z} \end{bmatrix} = \begin{bmatrix} \gamma_1\gamma_2 + \gamma_1\gamma_2 v_{1x} v_2 \\ \gamma_1\gamma_2 v_2 + \gamma_1\gamma_2 v_{1x} \\ \gamma_1 v_{1y} \\ \gamma_1 v_{1z} \end{bmatrix}$$

new frame: $\gamma = \gamma_1\gamma_2 + \gamma_1\gamma_2 v_{1x} v_2$

$\gamma v_x = \gamma_1\gamma_2 v_2 + \gamma_1\gamma_2 v_{1x}$

$$v_x = \frac{\gamma_1\gamma_2 v_2 + \gamma_1\gamma_2 v_{1x}}{\gamma_1\gamma_2 + \gamma_1\gamma_2 v_{1x} v_2} = \boxed{\frac{v_2 + v_{1x}}{1 + v_{1x} v_2}} = v_x$$

$$\boxed{v_y = \frac{v_{1y}}{\gamma_2(1 + v_{1x} v_2)}}$$

$$\boxed{v_z = \frac{v_{1z}}{\gamma_2(1 + v_{1x} v_2)}}$$

✓

b) show if $\vec{v}_1 = v_{1x}\hat{x}$, $v = \frac{v_1 + v_2}{1 + v_1 v_2}$

$\therefore v_{1y} = v_{1z} = 0$

$u_1^\mu = \begin{pmatrix} \gamma_1 \\ \gamma_1 v_{1x} \\ 0 \\ 0 \end{pmatrix} \rightarrow u^\nu = \begin{bmatrix} \gamma_1\gamma_2(1 + v_1 v_2) \\ \gamma_1\gamma_2(v_2 + v_1) \\ 0 \\ 0 \end{bmatrix}$

$v = \sqrt{v_x^2 + v_y^2 + v_z^2} = v_x$ b/c $v_y = v_z = 0$

$$\boxed{v = \frac{v_2 + v_1}{1 + v_1 v_2}}$$ ✓

✓

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2. $\eta$ = "rapidity", $v = \tanh(\eta)$

$c = 1$, so $-1 \leq v \leq 1 \rightarrow -\infty < \eta < \infty$

(restricted to one spacial direction)

a) Show $u^\mu = \begin{bmatrix} \cosh(\eta) \\ \sinh(\eta) \\ 0 \\ 0 \end{bmatrix}$, $u^\mu = \begin{pmatrix} \gamma \\ \gamma v_x \\ \gamma v_y \\ \gamma v_z \end{pmatrix}$

b/c $v_y = v_z = 0$, $|\vec{v}| = v_x$

$\gamma = \frac{1}{\sqrt{1 - \vec{v}^2}}$ $\quad v^2 = \tanh^2(\eta)$

$1 - \tanh^2(\eta) = \operatorname{sech}^2 \eta$

$\gamma = \frac{1}{\operatorname{sech}^2 \eta} = \cosh \eta$

$v_x = \tanh \eta$, $\gamma v_x = \tanh \eta \cosh \eta = \sinh \eta$

$\therefore \boxed{u^\mu = \begin{bmatrix} \cosh \eta \\ \sinh \eta \\ 0 \\ 0 \end{bmatrix}}$ ✓

b) Show that while velocities do not add linearly, rapidities do.

$\left( v = \frac{v_1 + v_2}{1 + v_1 v_2} \right)$ $\quad v_1 = \tanh(\eta_1)$ $\quad v_2 = \tanh(\eta_2)$

$\eta = \eta_1 + \eta_2$

$v = \tan(\eta) = \tan(\eta_1 + \eta_2)$

Schaum's: $\tanh(x+y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$

$v = \frac{\tanh \eta_1 + \tanh \eta_2}{1 + \tanh \eta_1 \tanh \eta_2} = \frac{v_1 + v_2}{1 + v_1 v_2}$ ✓

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4. Rocket firing at constant rate
rest frame it experiences acceleration in +x direction

a) Show that EOM: $\frac{du^\mu}{d\tau} = G^{\mu\nu} u_\nu$, $G^{\mu\nu} = \begin{pmatrix} 0 & -g & 0 & 0 \\ g & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

1st show acceleration in comoving is correct

$$a^\mu = G^{\mu\nu} u_\nu = \begin{bmatrix} 0 & -g & 0 & 0 \\ g & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \gamma \\ -\gamma v_x \\ -\gamma v_y \\ -\gamma v_z \end{bmatrix} = \begin{bmatrix} g\gamma v_x \\ g\gamma \\ 0 \\ 0 \end{bmatrix} = g\gamma \begin{bmatrix} v_x \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Comoving frame
moving in propertime, $\frac{du^\mu}{d\tau} = \frac{du^\mu}{dt}$ (b/c $v_{rocket} = 0$ in comoving frame), $v \ll 1 \rightarrow u^{\mu'} = \begin{bmatrix} 1 \\ v_x \\ v_y \\ v_z \end{bmatrix}$ (such that $\gamma \sim 1$) $\rightarrow \frac{du^\mu}{dt} = \begin{bmatrix} 0 \\ a \\ 0 \\ 0 \end{bmatrix}$

$$G^{\mu\nu} u_\nu = \begin{bmatrix} 0 & -g & 0 & 0 \\ g & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ g \\ 0 \\ 0 \end{bmatrix}$$

so, $\frac{du^\mu}{d\tau} = G^{\mu\nu} u_\nu \rightarrow a = g$ in comoving frame ✓

$$\frac{d(u_\mu u^\mu)}{d\tau} = \left(\frac{du_\mu}{d\tau}\right) u^\mu + u_\mu \left(\frac{du^\mu}{d\tau}\right)$$

$$= a_\mu u^\mu + u_\mu a^\mu$$

$$a^\mu = \begin{bmatrix} g\gamma v_x \\ g\gamma \\ 0 \\ 0 \end{bmatrix}, \quad a_\mu = \begin{bmatrix} g\gamma v_x \\ -g\gamma \\ 0 \\ 0 \end{bmatrix}, \quad u^\mu = \begin{bmatrix} \gamma \\ \gamma v_x \\ \gamma v_y \\ \gamma v_z \end{bmatrix} \quad u_\mu = \begin{bmatrix} \gamma \\ -\gamma v_x \\ -\gamma v_y \\ -\gamma v_z \end{bmatrix}$$

$$\frac{d(u_\mu u^\mu)}{d\tau} = (g\gamma^2 v_x - g\gamma^2 v_x) + (\gamma^2 g v_x - \gamma^2 g v_x) = 0 \quad ✓$$

b) Starts at origin at $t=0$

find expression for $u^\mu(\tau)$

$\frac{du^\mu}{d\tau} = a^\mu$, split into components ↘

$$\frac{du^0}{d\tau} = a^0 = g u^1 \qquad \frac{du^2}{d\tau} = a^2 = 0$$

$$\frac{du^1}{d\tau} = a^1 = g u^0 \qquad \frac{du^3}{d\tau} = a^3 = 0$$

4 cont

$$\frac{d}{d\tau}\left(\frac{du^1}{d\tau} = gu^0\right) \Rightarrow \frac{d^2u^1}{d\tau^2} = g\frac{du^0}{d\tau} = g^2u^1$$

$$\frac{d^2u^1}{d\tau^2} - g^2u^1 = 0$$

has solutions of the form $u^1 = Ae^{g\tau} + Be^{-g\tau}$

OR

$u^1 = A\sinh g\tau + B\cosh g\tau$

because the rocket starts from rest at $t=0$ at origin ($\tau = 0$)

$$u^1(\tau=0) = A\sinh(0) + B\cosh(0) = B \quad \therefore \quad B=0$$

$$u^1(\tau) = A\sinh g\tau$$

$$gu^0 = \frac{du^1}{d\tau} = Ag\cosh g\tau \rightarrow u^0(\tau) = A\cosh g\tau$$

$u^2$ and $u^3$ = constants, but b/c of the I.C. $u^2 = u^3 = 0$

$$u^\mu = \begin{bmatrix} A\cosh g\tau \\ A\sinh g\tau \\ 0 \\ 0 \end{bmatrix}, \quad u^\mu u_\mu = A^2\cosh^2 g\tau - A^2\sinh^2 g\tau = A^2 = 1$$

$$A = \pm 1$$

$A = +1$ b/c the acceleration is in the $+x$ direction

$$\boxed{u^\mu(\tau) = \begin{bmatrix} \cosh g\tau \\ \sinh g\tau \\ 0 \\ 0 \end{bmatrix}}$$

✓

c) compute $x^\mu(\tau)$

$$\frac{dx^0}{d\tau} = \cosh g\tau \qquad \frac{dx^1}{d\tau} = \sinh g\tau \qquad \frac{dx^2}{d\tau} = \frac{dx^3}{d\tau} = 0$$

$$\downarrow \qquad\qquad \downarrow$$

$$x^0 = \frac{\sinh g\tau}{g} + c_0 \qquad x^1 = \frac{\cosh g\tau}{g} + c_1 \qquad x^2 = c_2 \qquad x^3 = c_3$$

4 cont

apply I.C.

$x^0(\tau=0) = 0 \rightarrow c_0 = 0$

$x^2(\tau=0) = 0 \rightarrow c_2 = 0$

$x^3(\tau=0) = 0 \rightarrow c_3 = 0$

$x^1(\tau=0) = 0 = \frac{1}{g} + c_1 \rightarrow c_1 = -\frac{1}{g}$

$$x^\mu(\tau) = \frac{1}{g}\begin{bmatrix} \sinh g\tau \\ \cosh g\tau - 1 \\ 0 \\ 0 \end{bmatrix}$$

d) Find $x(t)$

from part c, $t(\tau) = \frac{1}{g}\sinh g\tau$

$gt = \sinh g\tau$

$\tau = \frac{\sinh^{-1}(gt)}{g}$

$x(\tau) = \frac{1}{g}(\cosh g\tau - 1)$

$\downarrow$

$$x(t) = \frac{1}{g}\left(\cosh(\sinh^{-1}(gt)) - 1\right)$$

$= \frac{1}{g}\left[\sqrt{1+g^2t^2} - 1\right]$

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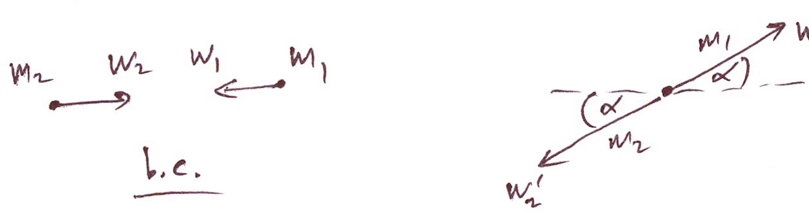
b.c. a.c.

**LAB**


b.c. a.c.

**CM**

**Define**

$v = \tanh\eta$ $\quad v_1' = \tanh\eta_1'$ $\quad v_1'\cos\theta_1 = \tanh\lambda_1$

$v_{cm} = \tanh\eta_{cm}$ $\quad v_2' = \tanh\eta_2'$ $\quad v_1'\sin\theta_1 = \tanh\bar{\lambda}_1$

$v_2'\cos\theta_2 = \tanh\lambda_2$

$v_2'\sin\theta_2 = \tanh\bar{\lambda}_2$

$w_1 = \tanh\xi_1$ $\quad w_1' = \tanh\xi_1'$ $\quad w_1'\cos\alpha = \tanh\mu_1$

$w_2 = \tanh\xi_2$ $\quad w_2' = \tanh\xi_2'$ $\quad w_1'\sin\alpha = \tanh\bar{\mu}_1$

$w_2'\cos\alpha = \tanh\mu_2$

$w_2'\sin\alpha = \tanh\bar{\mu}_2$

**Find $v_{cm}$**. **Set $c = 1$.**

Look the particles before collision. In lab frame, the total energy and total momentum are $E = m_1 + \gamma(v)m_2$ and $P = \gamma(v)m_2 v$. In CM frame, the total spatial momentum is zero. So, using Lorentz transformation,

$$0 = -\gamma(v_{cm})\,v_{cm}\,E + \gamma(v_{cm})\,P$$

$$v_{cm} = \frac{P}{E}$$

$$\boxed{v_{cm} = \frac{\gamma(v)\,m_2 v}{m_1 + \gamma(v)\,m_2}}$$

## Addition of velocities

$$\eta = \xi_2 + \eta_{cm}$$
$$0 = -\xi_1 + \eta_{cm}$$
} b.c.

$$\lambda_1 = \mu_1 + \eta_{cm}$$
$$\lambda_2 = -\mu_2 + \eta_{cm}$$
} a.c. z-direction

$$\bar{\lambda}_1 = \bar{\mu}_1$$
$$\bar{\lambda}_2 = \bar{\mu}_2$$
} a.c. transverse direction

## Find $w_1, w_2$.

$$w_1 = \tanh \xi_1 = \tanh \eta_{cm} = v_{cm}$$

$$\boxed{w_1 = \frac{\gamma(v) m_2 v}{m_1 + \gamma(v) m_2}}$$

$$w_2 = \tanh \xi_2 = \tanh(\eta - \eta_{cm}) = \frac{\tanh \eta - \tanh \eta_{cm}}{1 - \tanh \eta \cdot \tanh \eta_{cm}} = \frac{v - v_{cm}}{1 - v v_{cm}}$$

$$\boxed{w_2 = \frac{\gamma(v) m_1 v}{\gamma(v) m_1 + m_2}}$$

So basically we have written $w_1, w_2$ and $v_{cm}$ as the functions of $v$. Then it is sufficient if we express the next variables in terms of $w_1, w_2$ and $v_{cm}$.

Note that if $m_1 = m_2$, $w_1 = w_2$.

Find $w_1'$, $w_2'$.

We work in CM frame, and use conservation of energy and momentum.

$$\gamma(w_1)\, m_1 + \gamma(w_2)\, m_2 = \gamma(w_1')\, m_1 + \gamma(w_2')\, m_2$$

$$\gamma(w_1)\, m_1 w_1 = \gamma(w_2)\, m_2 w_2$$

$$\gamma(w_1') m_1 w_1' = \gamma(w_2')\, m_2 w_2'$$

In rapidity notation,

$$m_1 \cosh \xi_1 + m_2 \cosh \xi_2 = m_1 \cosh \xi_1' + m_2 \cosh \xi_2'$$

$$m_1 \sinh \xi_1 = m_2 \sinh \xi_2$$

$$m_1 \sinh \xi_1' = m_2 \sinh \xi_2'$$

The last two equations, however, can be written as

$$m_1 w_1 \cosh \xi_1 = m_2 w_2 \cosh \xi_2$$

$$m_1 w_1' \cosh \xi_1' = m_2 w_2' \cosh \xi_2'$$

So, subtituting to the first equation,

$$m_1 \cosh \xi_1 + m_1 \frac{w_1}{w_2} \cosh \xi_1 = m_1 \cosh \xi_1' + m_1 \frac{w_1'}{w_2'} \cosh \xi_1'$$

But,

$$\frac{w_1'}{w_2'} = \frac{m_2}{m_1} \frac{\cosh \xi_2'}{\cosh \xi_1'} = \frac{m_2}{m_1} \frac{\gamma(w_2')}{\gamma(w_1')} = \frac{m_2}{m_1} \sqrt{\frac{1-(w_1')^2}{1-(w_2')^2}}$$

$$\left(\frac{w_1'}{w_2'}\right)^2 = \left(\frac{m_2}{m_1}\right)^2 \left(\frac{1-(w_1')^2}{1-(w_2')^2}\right)$$

$$\left(\frac{w_1'}{w_2'}\right)^2 - (w_1')^2 = \left(\frac{m_2}{m_1}\right)^2 \left(1-(w_1')^2\right)$$

$$\left(\frac{w_1'}{w_2'}\right)^2 = (w_1')^2 + \left(\frac{m_2}{m_1}\right)^2 \left(1-(w_1')^2\right)$$

Substitute it back, we'll get

$$m_1 \cosh \xi_1 \left(1+\frac{w_1}{w_2}\right) = m_1 \cosh \xi_1' \left(1+\sqrt{(w_1')^2+\left(\frac{m_2}{m_1}\right)^2\left(1-(w_1')^2\right)}\right)$$

$$\gamma(w_1)\left(1+\frac{w_1}{w_2}\right) = \gamma(w_1') + \sqrt{\gamma(w_1')^2\left(1-\frac{1}{\gamma(w_1')^2}\right) + \left(\frac{m_2}{m_1}\right)^2}$$

$$\gamma(w_1)\left(1+\frac{w_1}{w_2}\right) = \gamma(w_1') + \sqrt{\gamma(w_1')^2 + \left[\left(\frac{m_2}{m_1}\right)^2 - 1\right]}$$

$$\gamma(w_1)\left(1+\frac{w_1}{w_2}\right) - \gamma(w_1') = \sqrt{\gamma(w_1')^2 + \left[\left(\frac{m_2}{m_1}\right)^2 - 1\right]}$$

$$\gamma(w_1)^2\left(1+\frac{w_1}{w_2}\right)^2 + \gamma(w_1')^2 - 2\gamma(w_1)\gamma(w_1')\left(1+\frac{w_1}{w_2}\right) = \gamma(w_1')^2 + \left[\left(\frac{m_2}{m_1}\right)^2 - 1\right]$$

$$\gamma(w_1)^2\left(1+\frac{w_1}{w_2}\right)^2 - \left[\left(\frac{m_2}{m_1}\right)^2 - 1\right] = 2\gamma(w_1)\gamma(w_1')\left(1+\frac{w_1}{w_2}\right)$$

$$\boxed{\gamma(w_1') = \frac{1}{2}\gamma(w_1)\left(1+\frac{w_1}{w_2}\right) - \frac{1}{2\gamma(w_1)}\left(1+\frac{w_1}{w_2}\right)^{-1}\left[\left(\frac{m_2}{m_1}\right)^2 - 1\right]}$$

Note that if $m_1 = m_2$, $\gamma(w_1') = \gamma(w_1)$.

Back to our first equation (the conservation of energy),

$$\gamma(w_1)\,m_1 + \gamma(w_2)\,m_2 = \gamma(w_1')\,m_1 + \gamma(w_2')\,m_2$$

$$= \frac{1}{2}m_1\gamma(w_1)\left(1+\frac{w_1}{w_2}\right) - \frac{m_1}{2\gamma(w_1)}\left(1+\frac{w_1}{w_2}\right)^{-1}\left[\left(\frac{m_2}{m_1}\right)^2 - 1\right]$$

$$+ \gamma(w_2')\,m_2$$

$$\frac{m_1}{m_2}\gamma(w_1) + \gamma(w_2) = \frac{1}{2}\frac{m_1}{m_2}\gamma(w_1)\left(1+\frac{w_1}{w_2}\right) - \frac{1}{2\gamma(w_1)}\frac{m_1}{m_2}\left[\left(\frac{m_2}{m_1}\right)^2 - 1\right]\left(1+\frac{w_1}{w_2}\right)^{-1} + \gamma(w_2')$$

$$\boxed{\gamma(w_2') = \frac{1}{2}\frac{m_1}{m_2}\gamma(w_1)\left(1-\frac{w_1}{w_2}\right) + \gamma(w_2) + \frac{1}{2\gamma(w_1)}\frac{m_1}{m_2}\left[\left(\frac{m_1}{m_2}\right)^2 - 1\right]\left(1+\frac{w_1}{w_2}\right)^{-1}}$$

If $m_1 = m_2$, $\gamma(w_2') = \gamma(w_2)$.

Find $v_1'$.

$$v_1'\sin\theta_1 = \tanh\bar{\lambda}_1 = \tanh\bar{\mu}_1 = w_1'\sin\alpha \implies \sin\alpha = \frac{v_1'}{w_1'}\sin\theta_1$$

$$v_1'\cos\theta_1 = \tanh\lambda_1 = \tanh(\mu_1 + \eta_{cm}) = \frac{\tanh\mu_1 + \tanh\eta_{cm}}{1+\tanh\mu_1\cdot\tanh\eta_{cm}}$$

$$= \frac{w_1'\cos\alpha + v_{cm}}{1 + w_1' v_{cm}\cos\alpha}$$

But, $\sin^2\alpha = \left(\frac{V_1'}{W_1'}\right)^2 \sin^2\theta_1$, so $(W_1')^2\cos^2\alpha = (W_1')^2 - (V_1')^2\sin^2\theta_1$.

It implies

$$V_1'\cos\theta_1 = \frac{\sqrt{(W_1')^2-(V_1')^2\sin^2\theta_1} + V_{cm}}{1+V_{cm}\sqrt{(W_1')^2-(V_1')^2\sin^2\theta_1}}$$

$$V_1'\cos\theta_1 + V_{cm}V_1'\cos\theta_1\sqrt{(W_1')^2-(V_1')^2\sin^2\theta_1} = V_{cm} + \sqrt{(W_1')^2-(V_1')^2\sin^2\theta_1}$$

$$V_1'\cos\theta_1 - V_{cm} = (1-V_{cm}V_1'\cos\theta_1)\sqrt{(W_1')^2-(V_1')^2\sin^2\theta_1}$$

Squaring this equation, we'll get

$$\left[V_{cm}^2\sin^2\theta_1\cos^2\theta_1\right](V_1')^4 - \left[2V_{cm}\cos\theta_1\sin^2\theta_1\right](V_1')^3 + \left[1-V_{cm}^2(W_1')^2\cos^2\theta_1\right](V_1')^2 + \left[2V_{cm}\cos\theta_1\left((W_1')^2-1\right)\right]V_1' + \left[V_{cm}^2-(W_1')^2\right] = 0$$

We can find $V_1'$ (principally) from this polynomial.

We also can claim that we can get $\alpha$ from equation

$$\sin\alpha = \frac{V_1'}{W_1'}\sin\theta_1$$

where $V_1'$ and $W_1'$ are functions of $v$ and $\theta_1$.

Find $\theta_2$ & $V_2'$.

$$V_2'\sin\theta_2 = \tanh\bar{\lambda}_2 = \tanh\bar{\mu}_2 = W_2'\sin\alpha$$

$$V_2'\cos\theta_2 = \tanh\lambda_2 = \tanh(-\mu_2+\eta_{cm}) = \frac{\tanh\eta_{cm}-\tanh\mu_2}{1-\tanh\eta_{cm}\cdot\tanh\mu_2}$$

$$= \frac{V_{cm}-W_2'\cos\alpha}{1-V_{cm}W_2'\cos\alpha}$$

So, $$\tan\theta_2 = W_2'\sin\alpha\cdot\frac{1-V_{cm}W_2'\cos\alpha}{V_{cm}-W_2'\cos\alpha}$$

where $W_2'$, $\alpha$ and $V_{cm}$ are functions of $v$ and $\theta_1$.

$$V_2' = W_2'\frac{\sin\alpha}{\sin\theta_2}$$

where $W_2'$, $\alpha$ and $\theta_2$ are functions of $v$ and $\theta_1$.