Homework 12 - Solutions

Problem 1

a)

The Lagrangian is given by $L = \frac{1}{2}I\dot{\theta}^2 + \frac{V_0}{\lambda}\cos(\lambda^{1/2}\theta)$. Since θ is small, we can expand the cosine,

$$\begin{aligned} L &= \frac{1}{2}I\dot{\theta}^2 + \frac{V_0}{\lambda}\Big(1 - \frac{(\lambda^{1/2}\theta)^2}{2!} + \frac{(\lambda^{1/2}\theta)^4}{4!} - \frac{(\lambda^{1/2}\theta)^6}{6!} + \cdots\Big) \\ &= \frac{V_0}{\lambda} + \frac{1}{2}I\dot{\theta}^2 - V_0\frac{\theta^2}{2!} + \lambda V_0\frac{\theta^4}{4!} - \lambda^2 V_0\frac{\theta^6}{6!} + \cdots \end{aligned} \tag{1}$$

Apart from the first term on the RHS which has no effect to the equation of motion, we can see that the Lagrangian now is the summation of leading order, perturbation and smaller perturbation term.

b)

The equation of motion can be obtained from

$$\frac{d}{dt}\Big(\frac{\partial L}{\partial\dot{\theta}}\Big) - \frac{\partial L}{\partial\theta} = 0 \tag{2}$$

and it gives

$$I\ddot{\theta} + \frac{V_0}{\lambda^{1/2}}\sin(\lambda^{1/2}\theta) = 0 \tag{3}$$

Expand the sine function because θ is small,

$$\sin(\lambda^{1/2}\theta) = \lambda^{1/2}\theta - \frac{(\lambda^{1/2}\theta)^3}{3!} + \frac{(\lambda^{1/2}\theta)^5}{5!} + \cdots \tag{4}$$

We can write θ as $\theta = \theta_0 + \lambda\theta_1 + \lambda^2\theta_2 + \cdots$, hence $\ddot{\theta} = \ddot{\theta}_0 + \lambda\ddot{\theta}_1 + \lambda^2\ddot{\theta}_2 + \cdots$, such that we will have

$$I(\ddot{\theta}_0 + \lambda\ddot{\theta}_1 + \lambda^2\ddot{\theta}_2 + \cdots) = -V_0\Big\{\theta_0 + \lambda\Big(\theta_1 - \frac{\theta_0}{3!}\Big) + \lambda^2\Big(\theta_2 - \frac{\theta_1}{2!} + \frac{\theta_0}{5!}\Big) + \cdots\Big\} \tag{5}$$

Equating the coefficient of λ, we will have

$$\ddot{\theta}_0 = -\frac{V_0}{I}\theta_0 \tag{6}$$

$$\ddot{\theta}_1 = -\frac{V_0}{I}\Big(\theta_1 - \frac{\theta_0^3}{3!}\Big) \tag{7}$$

$$\ddot{\theta}_2 = -\frac{V_0}{I}\Big(\theta_2 - \frac{\theta_0^2\theta_1}{2!} + \frac{\theta_0^5}{5!}\Big) \tag{8}$$

And the last computation can be done using Mathematica to get

$$\theta_0(t) = \theta_0(0)\cos(\Omega t) \tag{9}$$

$$\theta_1(t) = \frac{1}{192}\Big\{\Big(\theta_0^3(0)+192\theta_1(0)\Big)\cos(\Omega t) -\theta_0^3(0)\Big(\cos(3\Omega t)-12\Omega t\sin(\Omega t)\Big)\Big\} \tag{10}$$

$$\begin{aligned}\theta_2(t) = {} & \frac{1}{61440}\Big\{\Big(960\theta_0^2(0)\theta_1(0)+61440\theta_2(0)+\theta_0^5(0)(17-120\Omega^2t^2)\Big)\cos(\Omega t) \\ & +\theta_0^2(0)\Big(-20(\theta_0^3(0)+48\theta_1(0))\cos(3\Omega t)+3(3840\theta_1(0)\Omega t\sin(\Omega t) \\ & +\theta_0^3(0)(\cos(5\Omega t)-20\Omega t\sin(3\Omega t)))\Big)\Big\}\end{aligned} \tag{11}$$

where $\Omega \equiv V_0/I$.

Problem 2

We write θ as

$$\theta(t) = \sum_n c_n \cos\left((2n+1)\omega t\right) \tag{12}$$

where

$$c_n = \sum_i \lambda^{n+i} c_n^{(i)}, \qquad \omega = \sum_i \lambda^i \omega_i \tag{13}$$

so,

$$\begin{aligned}
\theta(t) &= \sum_{n,i} \lambda^{n+i} c_n^{(i)} \cos\Big((2n+1)\sum_j \lambda^j \omega_j t\Big) \\
&= \sum_{n,i} \lambda^{n+i} c_n^{(i)} \sum_k \frac{(-1)^k}{(2k)!}\Big((2n+1)\sum_j \lambda^j \omega_j t\Big)^{2k} \\
&= \sum_{n,i,k} \lambda^{n+i} c_n^{(i)} \frac{(-1)^k}{(2k)!}(2n+1)^{2k} t^{2k}\Big(\sum_j \lambda^j \omega_j\Big)^{2k} \\
&\approx \sum_{n,i,k} \frac{(-1)^k}{(2k)!}(2n+1)^{2k} c_n^{(i)} \lambda^{n+i} \omega_0^{2k-1} t^{2k}\Big(\omega_0 + 2k\lambda\omega_1 + 2k\lambda^2\omega_2 + \cdots\Big)
\end{aligned} \tag{14}$$

The zeroth order of λ can be obtained by settng $n = i = 0$, with the relevant term in the bracket comes only from the zeroth order in λ. The first order can be obtained by setting $n = i = 0$ with the relevant term in the bracket is the first order, and $n = 0, i = 1$ and $n = 1, i = 0$ with the relevant term in the bracket is the zeroth order. And so on. Therefore, we will get

$$
\begin{aligned}
\theta(t) &= c_0^{(0)} \sum_k \frac{(-1)^k}{(2k)!} \omega_0^{2k} t^{2k} \\
&+\lambda\Big(c_0^{(0)} \omega_1 t \sum_k \frac{(-1)^k}{(2k)!} 2k \omega_0^{2k-1} t^{2k-1} + c_0^{(1)} \sum_k \frac{(-1)^k}{(2k)!} \omega_0^{2k} t^{2k} \\
&+ c_1^{(0)} \sum_k \frac{(-1)^k}{(2k)!} 3^{2k} \omega_0^{2k} t^{2k} \Big) \\
&+\lambda^2 \Big(c_0^{(0)} \omega_2 t \sum_k \frac{(-1)^k}{(2k)!} 2k \omega_0^{2k-1} t^{2k-1} \\
&+ c_0^{(1)} \omega_1 t \sum_k \frac{(-1)^k}{(2k)!} 2k \omega_0^{2k-1} t^{2k-1} + c_0^{(2)} \sum_k \frac{(-1)^k}{(2k)!} \omega_0^{2k} t^{2k} \\
&+ c_1^{(0)} \omega_1 t \sum_k \frac{(-1)^k}{(2k)!} 2k \omega_0^{2k-1} t^{2k-1} + c_1^{(1)} \sum_k \frac{(-1)^k}{(2k)!} 3^{2k} \omega_0^{2k} t^{2k} \\
&+ c_2^{(0)} \sum_k \frac{(-1)^k}{(2k)!} 5^{2k} \omega_0^{2k} t^{2k} \\
&+ \cdots \qquad (15) \\
&= c_0^{(0)} \cos(\omega_0 t) \\
&+\lambda\Big(- c_0^{(0)} \omega_1 t \sin(\omega_0 t) + c_0^{(1)} \cos(\omega_0 t) + c_1^{(0)} \cos(3\omega_0 t) \Big) \\
&+\lambda^2 \Big(- c_0^{(0)} \omega_2 t \sin(\omega_0 t) - c_0^{(1)} \omega_1 t \sin(\omega_0 t) + c_0^{(2)} \cos(\omega_0 t) \\
&- 3 c_1^{(0)} \sin(3\omega_0 t) + c_1^{(1)} \cos(3\omega_0 t) + c_2^{(0)} \cos(5\omega_0 t) \Big) \\
&+ \cdots \qquad (16)
\end{aligned}
$$

It implies

$$\theta_0 = c_0^{(0)} \cos(\omega_0 t) \qquad (17)$$

$$\theta_1 = -c_0^{(0)} \omega_1 t \sin(\omega_0 t) + c_0^{(1)} \cos(\omega_0 t) + c_1^{(0)} \cos(3\omega_0 t) \qquad (18)$$

$$
\begin{aligned}
\theta_2 &= -c_0^{(0)} \omega_2 t \sin(\omega_0 t) - c_0^{(1)} \omega_1 t \sin(\omega_0 t) + c_0^{(2)} \cos(\omega_0 t) \\
&\quad -3 c_1^{(0)} \omega_1 t \sin(3\omega_0 t) + c_1^{(1)} \cos(3\omega_0 t) + c_2^{(0)} \cos(5\omega_0 t) \qquad (19)
\end{aligned}
$$

By comparing these equations with what we got in problem 1, we will get

$$\begin{aligned}
\omega_0 &= \Omega && (20)\\
\omega_1 &= \frac{1}{16}\theta_0^2(0)\omega_0 && (21)\\
\omega_2 &= -\Big(\frac{1}{31072}\theta_0^4(0) + \frac{1}{4}\theta_1(0)\theta_0(0)\Big)\omega_0 && (22)
\end{aligned}$$

which implies

$$\omega = \omega_0\Big(1 - \frac{1}{4}\theta_1(0)\theta_0(0) + \frac{1}{16}\theta_0^2(0) - \frac{1}{3072}\theta_0^4(0)\Big) \tag{23}$$

where we set $\lambda = 1$.

Problem 3

a)

The velocities can be computed as

$$
\begin{aligned}
v_x &= \frac{p_x}{m} && (24)\\
v_y &= \frac{p_y - qB(t)x}{m} && (25)\\
v_z &= \frac{p_z}{m} && (26)
\end{aligned}
$$

b)

The Hamiltonian is given by

$$H = \frac{p_x^2 + (p_y - qB(t)x)^2 + p_z^2}{2m} \tag{27}$$

Since $H = \alpha$, then we can write

$$\frac{p_x^2}{2m\alpha - p_z^2} + \frac{q^2B^2(t)}{2m\alpha - p - z^2}\Big(x - \frac{p_y}{qB(t)}\Big)^2 = 1 \tag{28}$$

The area of the ellipse, which is determined by equation above, in the phase space (x, p_x) is J, so

$$J = \pi\frac{2m\alpha - p_z^2}{qB(t)} = \frac{\pi p_x^2}{qB(t)} + qB(t)\pi\Big(x - \frac{p_y}{qB(t)}\Big)^2 \tag{29}$$

It also implies

$$H = \frac{p_z^2}{2m} + \frac{JqB(t)}{2\pi m} \tag{30}$$

We can find W first to know what ω is. Indeed, we have

$$\frac{1}{2m}\Big(\Big(\frac{\partial W}{\partial x}\Big)^2 + (p_y - qB(t)x)^2 + p_z^2\Big) = \alpha \tag{31}$$

So, integrate equation above (using Mathematica), we will get

$$
\begin{aligned}
W &= \frac{1}{2qB(t)}\Big\{ -(p_y - qB(t)x)\sqrt{2m\alpha - p_z^2 - (p_y - qB(t)x)^2} \\
&\quad +(2m\alpha - p_z^2)\arctan\Big(-\frac{p_y - qB(t)x}{\sqrt{2m\alpha - p_z^2 - (p_y - qB(t)x)^2}}\Big)\Big\} && (32)\\
&= \frac{1}{2qB(t)}\Big\{ -(p_y - qB(t)x)\sqrt{\frac{JqB(t)}{\pi} - (p_y - qB(t)x)^2} \\
&\quad +\frac{JqB(t)}{\pi}\arctan\Big(-\frac{p_y - qB(t)x}{\sqrt{\frac{JqB(t)}{\pi} - (p_y - qB(t)x)^2}}\Big)\Big\} && (33)
\end{aligned}
$$

Since $\omega = \partial W/\partial J$, then by using Mathematica, we have

$$\omega = \frac{1}{2\pi} \arctan\left(\frac{p_y - qB(t)x}{p_x}\right) \tag{34}$$

c)

The adiabatic invariance theorem tells us that under slow variation of magnetic field, J is adiabatic invariant. So,

$$\frac{v^2 - v_z^2}{B} = \frac{p_x^2 + (p_y - qB(t)x)^2}{m^2 B} = \frac{Jq}{\pi} \tag{35}$$

is also adiabatic invariant.

d)

When the particle moves in a circular orbit of radius r, then

$$\frac{m(v^2 - v_z^2)}{r} = q\sqrt{v^2 - v_z^2}B \tag{36}$$

or,

$$\frac{v^2 - v_z^2}{B} = \frac{q^2 B r^2}{m} \tag{37}$$

Since the LHS is adiabatic invariant, then we conclude that the magnetic flux $\Phi = B\pi r^2$ is also adiabatic invariant.

e)

We have $B_0 \pi r_0^2 = 2B_0 \pi r^2$, so we get $r = r_0/\sqrt{2}$.