

Homework 10

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Note: Goldstein problems 5.8 and 5.14 are solved at the end.

Problem 1

(a)

The coefficients of the moment of inertia tensor in the usual 3×3 matrix representation are given by

$$I_{jk} = \int_V \rho(\mathbf{r})(r^2 \delta_{jk} - x_j x_k) dV \quad (1)$$

Therefore, for the given mass density, we have to evaluate the integral,

$$I_{jk} = \rho_0 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx dy dz e^{-\frac{x^2+y^2+z^2+xy}{2l^2}} (r^2 \delta_{jk} - x_j x_k) \quad (2)$$

The argument of the exponential is a quadratic form which can be written as

$$\frac{x^2 + y^2 + z^2 + xy}{2l^2} = \frac{1}{2} \mathbf{x}^T A \mathbf{x} \quad (3)$$

where

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad A = \begin{pmatrix} \frac{1}{l^2} & \frac{1}{2l^2} & 0 \\ \frac{1}{2l^2} & \frac{1}{l^2} & 0 \\ 0 & 0 & \frac{1}{l^2} \end{pmatrix} \quad (4)$$

so, A is real and symmetric. Define

$$F_{jk} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx dy dz e^{-\frac{1}{2} \mathbf{x}^T A \mathbf{x}} x_j x_k \quad (5)$$

and

$$G = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx dy dz e^{-\frac{1}{2} \mathbf{x}^T A \mathbf{x}} \quad (6)$$

In terms of F_{jk} , the moment of inertia coefficient I_{jk} is

$$I_{jk} = \rho_0 [(F_{11} + F_{22} + F_{33}) \delta_{jk} - F_{jk}] = \rho_0 [\text{tr}(F) \delta_{jk} - F_{jk}] \quad (7)$$

To find the coefficient, we use Wick's Theorem, stated below.

Wick's Theorem: If A is a real symmetric $N \times N$ matrix, and \mathbf{x} is an $N \times 1$ vector, then

$$\langle x_i x_j \cdots x_k x_l \rangle \triangleq \frac{\underbrace{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} dx_1 dx_2 \cdots dx_N e^{-\frac{1}{2} \mathbf{x}^T A \mathbf{x}} x_i x_j \cdots x_k x_l}_{N \text{ integrations}}}{\underbrace{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} dx_1 dx_2 \cdots dx_N e^{-\frac{1}{2} \mathbf{x}^T A \mathbf{x}}}_{N \text{ integrations}}} = \sum_{\sigma} (A^{-1})_{ab} \cdots (A^{-1})_{cd} \quad (8)$$

where the sum runs over all permutations $\sigma \equiv \{a, b, \cdots, c, d\}$ of the set of indices $\{i, j, \cdots, k, l\}$. ■

In particular, for $N = 3$, we have

$$\langle x_j x_k \rangle = \frac{F_{jk}}{G} = (A^{-1})_{jk} \quad (9)$$

To evaluate G , we use the following result on Gaussian integration.

Result: If A is a real symmetric $N \times N$ matrix, and \mathbf{x} is an $N \times 1$ vector, then

$$\underbrace{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} dx_1 dx_2 \cdots dx_N e^{-\frac{1}{2} \mathbf{x}^T A \mathbf{x}}}_{N \text{ integrations}} = \left(\frac{(2\pi)^N}{\det[A]} \right)^{1/2} \quad \blacksquare \quad (10)$$

For the matrix A defined in Eqn. (4), $\det(A) = \frac{3}{4l^6}$. For $N = 3$,

$$G = \sqrt{\frac{(2\pi)^3}{3/4l^6}} = 4\sqrt{\frac{2}{3}}\pi^{3/2}l^3 \quad (11)$$

From Eqns. (9) and (10), we get

$$F_{jk} = 4\sqrt{\frac{2}{3}}\pi^{3/2}l^3 \times (A^{-1})_{jk} \quad (12)$$

The inverse of A is

$$A^{-1} = \begin{pmatrix} \frac{4l^2}{3} & -\frac{2l^2}{3} & 0 \\ -\frac{2l^2}{3} & \frac{4l^2}{3} & 0 \\ 0 & 0 & l^2 \end{pmatrix} \quad (13)$$

So, using Eqn. (12), the matrix F is found to be

$$F = \begin{pmatrix} \frac{16}{3}\sqrt{\frac{2}{3}}\pi^{3/2}l^5 & -\frac{8}{3}\sqrt{\frac{2}{3}}\pi^{3/2}l^5 & 0 \\ -\frac{8}{3}\sqrt{\frac{2}{3}}\pi^{3/2}l^5 & \frac{16}{3}\sqrt{\frac{2}{3}}\pi^{3/2}l^5 & 0 \\ 0 & 0 & 4\sqrt{\frac{2}{3}}\pi^{3/2}l^5 \end{pmatrix} \quad (14)$$

and finally, using Eqn. (7), the matrix representation of the moment of inertia tensor is

$$\overleftrightarrow{I} = \begin{pmatrix} \frac{28\rho_0}{3}\sqrt{\frac{2}{3}}\pi^{3/2}l^5 & \frac{8\rho_0}{3}\sqrt{\frac{2}{3}}\pi^{3/2}l^5 & 0 \\ \frac{8\rho_0}{3}\sqrt{\frac{2}{3}}\pi^{3/2}l^5 & \frac{28\rho_0}{3}\sqrt{\frac{2}{3}}\pi^{3/2}l^5 & 0 \\ 0 & 0 & \frac{32\rho_0}{3}\sqrt{\frac{2}{3}}\pi^{3/2}l^5 \end{pmatrix} \quad (15)$$

(b)

The moment of inertia matrix obtained above is of the form

$$I = \begin{pmatrix} a & b & 0 \\ b & a & 0 \\ 0 & 0 & c \end{pmatrix} \quad (16)$$

where $a = \frac{28\rho_0}{3}\sqrt{\frac{2}{3}}\pi^{3/2}l^5$, $b = \frac{8\rho_0}{3}\sqrt{\frac{2}{3}}\pi^{3/2}l^5$ and $c = \frac{32\rho_0}{3}\sqrt{\frac{2}{3}}\pi^{3/2}l^5$. If λ is an eigenvalue, then it satisfies the secular equation,

$$|I - \lambda\mathbb{I}_3| = 0 \quad (17)$$

which is

$$(c - \lambda)(\lambda^2 - 2a\lambda + a^2 - b^2) = 0 \quad (18)$$

the solutions to which are $\lambda = a - b, a + b, c$. The eigenvector matrix V is given by

$$V = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (19)$$

So, the diagonal form of the moment of inertia tensor, which is the principal moment of inertia tensor, is given by

$$\overleftrightarrow{I}_D = V\overleftrightarrow{I}V^{-1} = \begin{pmatrix} a - b & 0 & 0 \\ 0 & a + b & 0 \\ 0 & 0 & c \end{pmatrix} \quad (20)$$

So, the principal moment of inertia tensor is

$$\overleftrightarrow{I}_D = \begin{pmatrix} \frac{20\rho_0}{3}\sqrt{\frac{2}{3}}\pi^{3/2}l^5 & 0 & 0 \\ 0 & 4\rho_0\sqrt{6}\pi^{3/2}l^5 & 0 \\ 0 & 0 & \frac{32\rho_0}{3}\sqrt{\frac{2}{3}}\pi^{3/2}l^5 \end{pmatrix} \quad (21)$$

Problem 2

For torque-free motion, Euler's equations in the principal axes frame are

$$I_1\dot{\omega}_1 = \omega_2\omega_3(I_2 - I_3) \quad (22)$$

$$I_2\dot{\omega}_2 = \omega_3\omega_1(I_3 - I_1) \quad (23)$$

$$I_3\dot{\omega}_3 = \omega_1\omega_2(I_1 - I_2) \quad (24)$$

The energy is

$$E = \frac{J_1^2}{2I_1} + \frac{J_2^2}{2I_2} + \frac{J_3^2}{2I_3} = \frac{1}{2}I_1\dot{\omega}_1^2 + \frac{1}{2}I_2\dot{\omega}_2^2 + \frac{1}{2}I_3\dot{\omega}_3^2 \quad (25)$$

So,

$$\begin{aligned} \frac{dE}{dt} &= \omega_1 I_1 \dot{\omega}_1 + \omega_2 I_2 \dot{\omega}_2 + \omega_3 I_3 \dot{\omega}_3 \\ &= \omega_1 [\omega_2 \omega_3 (I_2 - I_3)] + \omega_2 [\omega_3 \omega_1 (I_3 - I_1)] + \omega_3 [\omega_1 \omega_2 (I_1 - I_2)] \quad (\text{using Euler's equations}) \\ &= \omega_1 \omega_2 \omega_3 (I_2 - I_3 + I_3 - I_1 + I_1 - I_2) \\ &= 0 \end{aligned} \quad (26)$$

Therefore, the energy is constant.

Problem 3

(a)

Using the explicit form of the generators,

$$M_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (27)$$

we can identify the $(j, k)^{th}$ element of M_i as

$$(M_i)_{jk} = -\epsilon_{ijk} \quad (28)$$

Now

$$\begin{aligned} \Omega_{jk} &= -\boldsymbol{\omega} \cdot \overleftrightarrow{M} \\ &= -\omega_l (M_l)_{jk} \\ &= \omega_l \epsilon_{ljk} \end{aligned} \quad (29)$$

Contracting both sides of Eqn. (29) with ϵ_{ijk} and using the identity

$$\epsilon_{ijk} \epsilon_{ljk} = 2\delta_{il} \quad (30)$$

we get

$$\boxed{\omega_i = \frac{1}{2} \epsilon_{ijk} \Omega_{jk}} \quad (31)$$

as required.

(b)

The Levi-Civita symbol can be written as

$$\epsilon_{ijk} = \hat{e}_i \cdot (\hat{e}_j \times \hat{e}_k) \quad (32)$$

Suppose \overleftrightarrow{R} denotes the rotation matrix for an active rotation. This means that under rotation, a vector ω transforms according to the rule

$$\omega \longrightarrow \omega' = \overleftrightarrow{R} \omega \quad (33)$$

whereas the base vectors of the space transform according to the rule

$$\hat{e}_i \longrightarrow \hat{e}'_i = \overleftrightarrow{R}^T \hat{e}_i \quad (34)$$

That is, the effect of an active rotation on a vector is to rotate the base vectors passively. This means that under rotation, the Levi-Civita symbol transforms according to the rule

$$\epsilon_{ijk} \longrightarrow \epsilon'_{ijk} = (\overleftrightarrow{R}^T \hat{e}_i) \cdot ((\overleftrightarrow{R}^T \hat{e}_j) \times (\overleftrightarrow{R}^T \hat{e}_k)) \quad (35)$$

Now,

$$(\overleftrightarrow{R}^T \hat{e}_i)_j = \overleftrightarrow{R}_{jm}^T (\hat{e}_i)_m \quad (36)$$

$$= \overleftrightarrow{R}_{jm}^T \delta_{i,m} \quad (37)$$

$$= \overleftrightarrow{R}_{ji}^T \quad (38)$$

Therefore,

$$\epsilon'_{ijk} = (\overleftrightarrow{R}^T \hat{e}_i) \cdot ((\overleftrightarrow{R}^T \hat{e}_j) \times (\overleftrightarrow{R}^T \hat{e}_k)) \quad (\text{from Eqn. (35)}) \quad (39)$$

$$= (\overleftrightarrow{R}^T \hat{e}_i)_l \left[(\overleftrightarrow{R}^T \hat{e}_j) \times (\overleftrightarrow{R}^T \hat{e}_k) \right]_l \quad (\text{sum on } l) \quad (40)$$

$$= \overleftrightarrow{R}_{li}^T \left[\epsilon_{lrs} (\overleftrightarrow{R}^T \hat{e}_j)_r (\overleftrightarrow{R}^T \hat{e}_k)_s \right] \quad (\text{using Eqn. (38) and cross product defn.}) \quad (41)$$

$$= \overleftrightarrow{R}_{li}^T \left[\epsilon_{lrs} \overleftrightarrow{R}_{rj}^T \overleftrightarrow{R}_{sk}^T \right] \quad (\text{using Eqn. (38)}) \quad (42)$$

$$= \epsilon_{lrs} \overleftrightarrow{R}_{il}^T \overleftrightarrow{R}_{jr}^T \overleftrightarrow{R}_{ks}^T \quad (43)$$

That is, under a rotation, the Levi-Civita symbol transforms like a tensor of rank 3:

$$\epsilon_{ijk} \longrightarrow \epsilon'_{ijk} = \overleftrightarrow{R}_{il}^T \overleftrightarrow{R}_{jr}^T \overleftrightarrow{R}_{ks}^T \epsilon_{lrs} \quad (44)$$

(b)

By definition,

$$\Omega_{mn} = \epsilon_{mni} \omega_i \quad (45)$$

So, under rotation,

$$\Omega_{mn} \longrightarrow \Omega'_{mn} = \epsilon'_{mni} \omega'_i \quad (46)$$

$$= \underbrace{(\epsilon_{lrs} \overleftrightarrow{R}_{ml} \overleftrightarrow{R}_{nr} \overleftrightarrow{R}_{is})}_{\epsilon'_{mni}} \underbrace{(\overleftrightarrow{R}_{ij} \omega_j)}_{\omega'_i} \quad (\text{using Eqn. 44}) \quad (47)$$

$$= \epsilon_{lrs} \overleftrightarrow{R}_{ml} \overleftrightarrow{R}_{nr} \underbrace{(\overleftrightarrow{R}_{si}^T \overleftrightarrow{R}_{ij})}_{(\overleftrightarrow{R}^T \overleftrightarrow{R})_{sj} = \delta_{sj}} \omega_j \quad (\text{using orthogonality of } \overleftrightarrow{R}) \quad (48)$$

$$= \overleftrightarrow{R}_{ml} \overleftrightarrow{R}_{nr} [\epsilon_{lrs} \omega_s] \quad (49)$$

$$= \overleftrightarrow{R}_{ml} \overleftrightarrow{R}_{nr} \Omega_{lr} \quad (\text{using Eqn. (45)}) \quad (50)$$

So under rotation, Ω_{ij} indeed transforms like a tensor of rank 2:

$$\Omega_{ij} \longrightarrow \Omega'_{ij} = \overleftrightarrow{R}_{il} \overleftrightarrow{R}_{jm} \Omega_{lm} \quad (51)$$

Problem 4

For torque-free motion with $I_1 = I_2 \neq I_3$, Euler's equations are

$$I_1 \dot{\omega}_1 = \omega_2 \omega_3 (I_2 - I_3) \quad (52)$$

$$I_2 \dot{\omega}_2 = \omega_3 \omega_1 (I_3 - I_1) \quad (53)$$

$$I_3 \dot{\omega}_3 = \omega_1 \omega_2 (I_1 - I_2) = 0 \quad (54)$$

The third Euler equation implies $\dot{\omega}_3 = 0$ and hence

$$\omega_3(t) = \omega_3(0) = \omega_3^{(0)} = \text{constant} \quad (55)$$

Differentiating the first Euler equation with respect to time, we get

$$\begin{aligned} I_1 \ddot{\omega}_1 &= \dot{\omega}_2 \omega_3 (I_2 - I_3) + \omega_2 \dot{\omega}_3 (I_3 - I_2) \\ &= \dot{\omega}_2 \omega_3 (I_2 - I_3) \quad (\because \dot{\omega}_3 = 0) \\ &= \omega_1 \omega_3^2 \frac{(I_3 - I_1)(I_2 - I_3)}{I_2} \quad (\text{using the second Euler equation}) \end{aligned} \quad (56)$$

$$\begin{aligned} &= -\omega_1 \frac{(\omega_3^{(0)})^2 (I_1 - I_3)^2}{I_1} \quad (\because \omega_3 = \omega_3^{(0)} \text{ and } I_1 = I_2) \\ &= -\omega_1 k^2 I_1 \quad \text{where } k \triangleq \frac{|I_1 - I_3| |\omega_3^{(0)}|}{I_1} \end{aligned} \quad (57)$$

which is a second order differential equation in ω_1 . Similarly, differentiating the second Euler equation yields a second order differential equation in ω_2 . The two equations are

$$\ddot{\omega}_1 + k^2 \omega_1 = 0 \quad (58)$$

$$\ddot{\omega}_2 + k^2 \omega_2 = 0 \quad (59)$$

where

$$k \triangleq \frac{|I_1 - I_3|}{I_1} |\omega_3^{(0)}| \quad (60)$$

In case $\omega_3^{(0)} = 0$, we can infer from the first two Euler equations that $\omega_1(t) = \text{constant}$ and $\omega_2(t) = \text{constant}$, a degenerate case. We assume therefore that $\omega_3^{(0)}$ is strictly positive. In this case, the general solutions to Eqns. (58) and (59) are

$$\omega_1(t) = A_1 \cos(kt) + B_1 \sin(kt) \quad (61)$$

$$\omega_2(t) = A_2 \cos(kt) + B_2 \sin(kt) \quad (62)$$

$$(63)$$

The boundary conditions are

$$\omega_1(0) = \omega_1^{(0)}, \quad \dot{\omega}_1(0) = \frac{\omega_2^{(0)} \omega_3^{(0)} (I_1 - I_3)}{I_1} \quad (64)$$

$$\omega_2(0) = \omega_2^{(0)}, \quad \dot{\omega}_2(0) = -\frac{\omega_3^{(0)} \omega_1^{(0)} (I_1 - I_3)}{I_1} \quad (65)$$

where we have used the first two Euler equations to determine the time derivatives of the angular velocities at $t = 0$. So, the particular solutions are:

$$\omega_1(t) = \omega_1^{(0)} \cos(kt) + \frac{\omega_2^{(0)} \omega_3^{(0)} (I_1 - I_3)}{k I_1} \sin(kt)$$

$$\omega_2(t) = \omega_2^{(0)} \cos(kt) - \frac{\omega_3^{(0)} \omega_1^{(0)} (I_1 - I_3)}{k I_1} \sin(kt)$$

$$\omega_3(t) = \omega_3^{(0)}$$

which can be rewritten, using the definition of k (Eqn. 60) as

$$\omega_1(t) = \omega_1^{(0)} \cos(kt) + \omega_2^{(0)} \operatorname{sgn}(\omega_3^{(0)}) \operatorname{sgn}(I_1 - I_3) \sin(kt) \quad (66)$$

$$\omega_2(t) = \omega_2^{(0)} \cos(kt) - \omega_1^{(0)} \operatorname{sgn}(\omega_3^{(0)}) \operatorname{sgn}(I_1 - I_3) \sin(kt) \quad (67)$$

$$\omega_3(t) = \omega_3^{(0)} \quad (68)$$

where $\operatorname{sgn}(x)$ is the *signum function*, defined as

$$\operatorname{sgn}(x) = \begin{cases} +1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases} \quad (69)$$

In the degenerate case $I_1 = I_2 = I_3$, the quantity $k = 0$ and hence all angular velocities are constant, at their respective $t = 0$ values. However, in the general case, $I_1 = I_2 \neq I_3$, and we can infer that an asymmetric rigid body with exactly two equal principal moment of inertia coefficients can have uniform angular motion about exactly one principal axis. The angular motion of the body is such that the vector,

$$\boldsymbol{\omega}_{xy}(t) = \omega_1(t) \hat{x} + \omega_2(t) \hat{y} \quad (70)$$

where \hat{x} , \hat{y} and \hat{z} are now along the three directions of the principal axes and $\omega_1(t)$ and $\omega_2(t)$ are as defined by Eqns. (66) and (67), has a **constant magnitude** of $((\omega_1^{(0)})^2 + (\omega_2^{(0)})^2)$ and therefore its tip traces out a circle (as the vector itself rotates generating a cone) in the $x - y$ plane. The vector $\omega_3 \hat{z}$ therefore is along the normal to this plane. In other words, $\omega_{xy}(t)$ precesses around the third principal axis of the rigid body, with a frequency given of $k/2\pi$ revolutions per second.

A Frisbee can be considered to be an asymmetric disc with exactly two equal principal moment of inertia coefficients (say $I_x = I_y \neq I_z$). The above analysis tells us that if the initial momentum imparted by the person who throws it in the air is such that it in addition to a nonzero initial angular velocity along the \hat{z} principal axis, it also has a nonzero initial angular velocity along either the \hat{x} principal axis or the \hat{y} principal axis (or both), then its motion will not be a *pure spin* about the \hat{z} axis, but rather also include rotation about the two other principal axes, which will result in an overall wobble of the frisbee. The wobble can be minimized by ensuring that most of the initial angular momentum imparted goes into spin angular motion about the third principal axis (\hat{z} in our notation).

Problem 5

(a)

The rotation operator is $\overleftrightarrow{R} = \exp(-\hat{n} \cdot \overleftrightarrow{M}\phi)$ where

$$\overleftrightarrow{M} = \hat{x}M_1 + \hat{y}M_2 + \hat{z}M_3$$

Let $\vec{\Omega} = \hat{n}\phi$. Then,

$$d\overleftrightarrow{R} = -e^{-\phi\hat{n} \cdot \overleftrightarrow{M}} \hat{n} \cdot \overleftrightarrow{M} d\phi = -(d\vec{\Omega} \cdot \overleftrightarrow{M}) \overleftrightarrow{R}$$

and hence

$$\frac{d\overleftrightarrow{R}(t)}{dt} = -(\dot{\vec{\Omega}} \cdot \overleftrightarrow{M}) \overleftrightarrow{R} \quad (71)$$

$$= -(\vec{\omega}^{body}(t) \cdot \overleftrightarrow{M}) \overleftrightarrow{R}(t) \quad (72)$$

Integrating both sides wrt time between 0 and t , we get

$$\overleftrightarrow{R}(t) = \overleftrightarrow{R}(0) - \int_0^t dt' \vec{\omega}^{body}(t') \cdot \overleftrightarrow{M} \overleftrightarrow{R}(t') \quad (73)$$

(b)

Define the integral operator

$$\mathcal{L}(t_1, 0, t) \equiv \int_0^t dt_1 \vec{\omega}^{body}(t_1) \cdot \overleftrightarrow{M} \quad (74)$$

So, Eqn. (73) can be written as

$$\overleftrightarrow{R}(t) = \overleftrightarrow{R}(0) - \mathcal{L}(t_1, 0, t) \overleftrightarrow{R}(t_1) \quad (75)$$

$$\begin{aligned} &= \overleftrightarrow{R}(0) - \mathcal{L}(t_1, 0, t) [\overleftrightarrow{R}(0) - \mathcal{L}(t_2, 0, t_1) \overleftrightarrow{R}(t_2)] \\ &= \overleftrightarrow{R}(0) - \mathcal{L}(t_1, 0, t) \overleftrightarrow{R}(0) + \mathcal{L}(t_1, 0, t) \mathcal{L}(t_2, 0, t_1) \overleftrightarrow{R}(t_2) \end{aligned} \quad (76)$$

$$\begin{aligned} &= \overleftrightarrow{R}(0) - \mathcal{L}(t_1, 0, t) \overleftrightarrow{R}(0) + \mathcal{L}(t_1, 0, t) \mathcal{L}(t_2, 0, t_1) [\overleftrightarrow{R}(0) - \mathcal{L}(t_3, 0, t_2) \overleftrightarrow{R}(t_3)] \\ &= \overleftrightarrow{R}(0) - \mathcal{L}(t_1, 0, t) \overleftrightarrow{R}(0) + \mathcal{L}(t_1, 0, t) \mathcal{L}(t_2, 0, t_1) \overleftrightarrow{R}(0) - \mathcal{L}(t_1, 0, t) \mathcal{L}(t_2, 0, t_1) \mathcal{L}(t_3, 0, t_2) \overleftrightarrow{R}(t_3) \end{aligned} \quad (77)$$

$$\begin{aligned} &= \overleftrightarrow{R}(0) - \mathcal{L}(t_1, 0, t) \overleftrightarrow{R}(0) + \mathcal{L}(t_1, 0, t) \mathcal{L}(t_2, 0, t_1) \overleftrightarrow{R}(0) \\ &\quad - \mathcal{L}(t_1, 0, t) \mathcal{L}(t_2, 0, t_1) \mathcal{L}(t_3, 0, t_2) \overleftrightarrow{R}(0) + \mathcal{L}(t_1, 0, t) \mathcal{L}(t_2, 0, t_1) \mathcal{L}(t_3, 0, t_2) \mathcal{L}(t_4, 0, t_3) \overleftrightarrow{R}(t_4) \end{aligned} \quad (78)$$

$$= \overleftrightarrow{R}_0 + \overleftrightarrow{R}_1 + \overleftrightarrow{R}_2 + \overleftrightarrow{R}_3 + \dots \quad (79)$$

Comparing Eqns. (78) and (79) we get,

$$\overleftrightarrow{R}_0 = \overleftrightarrow{R}(0) \quad (80)$$

$$\overleftrightarrow{R}_1(t) = -\mathcal{L}(t_1, 0, t) \overleftrightarrow{R}(0) \quad (81)$$

$$\begin{aligned} \overleftrightarrow{R}_2(t) &= \mathcal{L}(t_1, 0, t) \mathcal{L}(t_2, 0, t_1) \overleftrightarrow{R}(0) \\ &= -\mathcal{L}(t_1, 0, t) (-\mathcal{L}(t_2, 0, t_1) \overleftrightarrow{R}(0)) \\ &= -\mathcal{L}(t_1, 0, t) \overleftrightarrow{R}_1(t_1) \end{aligned} \quad (82)$$

$$\begin{aligned} \overleftrightarrow{R}_3(t) &= -\mathcal{L}(t_1, 0, t) \mathcal{L}(t_2, 0, t_1) \mathcal{L}(t_3, 0, t_2) \overleftrightarrow{R}(0) \\ &= \mathcal{L}(t_1, 0, t) \mathcal{L}(t_2, 0, t_1) \overleftrightarrow{R}_1(t_2) \\ &= -\mathcal{L}(t_1, 0, t) \overleftrightarrow{R}_2(t_2) \end{aligned} \quad (83)$$

Proceeding in this fashion, we will obtain

$$\overleftrightarrow{R}_{n+1}(t) = -\mathcal{L}(t', 0, t) \overleftrightarrow{R}_n(t') \quad (84)$$

that is,

$$\overleftrightarrow{R}_{n+1}(t) = - \int_0^t dt' \vec{\omega}^{body}(t') \cdot \overleftrightarrow{M} \overleftrightarrow{R}_n(t') \quad (85)$$

as required.

(c)

The term $\overleftrightarrow{R}_2(t)$ has the integral representation

$$\begin{aligned} \overleftrightarrow{R}_2(t) &= \mathcal{L}(t_1, 0, t) \mathcal{L}(t_2, 0, t_1) \overleftrightarrow{R}(0) \\ &= \int_0^t dt' \int_0^{t'} dt'' [\vec{\omega}^{body}(t') \cdot \overleftrightarrow{M}] [\vec{\omega}^{body}(t'') \cdot \overleftrightarrow{M}] \overleftrightarrow{R}(0) \end{aligned} \quad (86)$$

Consider the figure shown below.

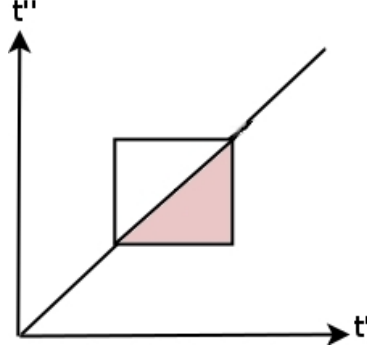


Diagram showing two regions of integration.

From the figure, it is evident that

$$\begin{aligned} \int_0^t dt' \int_0^{t'} dt'' &\longrightarrow \text{represents the lower (shaded) triangular area} \\ \int_0^t dt' \int_0^t dt'' &\longrightarrow \text{represents the entire square area} \end{aligned}$$

Therefore,

$$\begin{aligned} \overleftrightarrow{R}_2(t) &= \int_0^t dt' \int_0^{t'} dt'' [\vec{\omega}^{body}(t') \cdot \overleftrightarrow{M}] [\vec{\omega}^{body}(t'') \cdot \overleftrightarrow{M}] \overleftrightarrow{R}(0) \\ &= \frac{1}{2} T \left\{ \int_0^t dt' \int_0^t dt'' [\vec{\omega}^{body}(t') \cdot \overleftrightarrow{M}] [\vec{\omega}^{body}(t'') \cdot \overleftrightarrow{M}] \overleftrightarrow{R}(0) \right\} \end{aligned} \quad (87)$$

where T denotes the time ordering symbol. The term $\overleftrightarrow{R}_3(t)$ has the integral representation

$$\begin{aligned} \overleftrightarrow{R}_3(t) &= -\mathcal{L}(t_1, 0, t) \mathcal{L}(t_2, 0, t_1) \mathcal{L}(t_3, 0, t_2) \overleftrightarrow{R}(0) \\ &= - \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 [\vec{\omega}^{body}(t_1) \cdot \overleftrightarrow{M}] [\vec{\omega}^{body}(t_2) \cdot \overleftrightarrow{M}] [\vec{\omega}^{body}(t_3) \cdot \overleftrightarrow{M}] \overleftrightarrow{R}(0) \end{aligned}$$

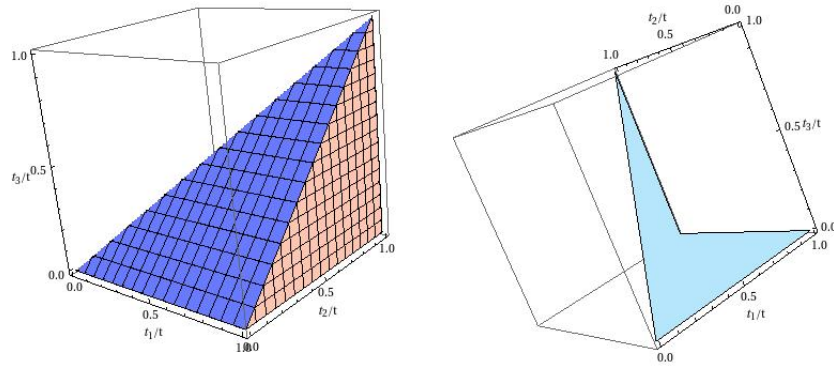
In three dimensions, the region

$$\mathcal{R} = \{(t_1, t_2, t_3) : 0 \leq t_1 \leq t, 0 \leq t_2 \leq t_1, 0 \leq t_3 \leq t_2, \text{ that is, } 0 \leq t_3 \leq t_2 \leq t_1 \leq t\} \quad (88)$$

defines a tetrahedron in 3D space with corners $(0, 0, 0)$, $(t, 0, 0)$, $(t, t, 0)$ and (t, t, t) .

The Mathematica command used to plot this region is:

```
RegionPlot3D[x >= 0 && y >= 0 && z >= 0 && x <= 1 && y <= x && z <= y,
             {x, 0, 1}, {y, 0, 1}, {z, 0, 1}, PlotPoints -> 100]
```



Plots of the tetrahedron representing the integration region, and two of its lateral faces.

The volume¹ of this tetrahedron is $\frac{1}{6}a^3$, that is, one-sixth the volume of the cube. Therefore, if the volume integral is carried out over the entire cube, we must divide by 6 or 3!. Hence,

$$\begin{aligned}
 \overleftrightarrow{R}_3(t) &= -\mathcal{L}(t_1, 0, t)\mathcal{L}(t_2, 0, t_1)\mathcal{L}(t_3, 0, t_2)\overleftrightarrow{R}(0) \\
 &= -\int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 [\vec{\omega}^{body}(t_1) \cdot \overleftrightarrow{M}][\vec{\omega}^{body}(t_2) \cdot \overleftrightarrow{M}][\vec{\omega}^{body}(t_3) \cdot \overleftrightarrow{M}]\overleftrightarrow{R}(0) \\
 &= -\frac{1}{3!}T \left\{ \int_0^t dt_1 \int_0^t dt_2 \int_0^t dt_3 [\vec{\omega}^{body}(t_1) \cdot \overleftrightarrow{M}][\vec{\omega}^{body}(t_2) \cdot \overleftrightarrow{M}][\vec{\omega}^{body}(t_3) \cdot \overleftrightarrow{M}]\overleftrightarrow{R}(0) \right\} \quad (89)
 \end{aligned}$$

Proceeding inductively in this fashion, we see that the n^{th} term in the expansion can be written in terms of a volume integral over the n -cube:

$$\overleftrightarrow{R}_n(t) = \frac{(-1)^n}{n!}T \left\{ \underbrace{\int_0^t dt_1 \dots \int_0^t dt_n}_{n \text{ integrations}} [\vec{\omega}^{body}(t_1) \cdot \overleftrightarrow{M}] \dots [\vec{\omega}^{body}(t_n) \cdot \overleftrightarrow{M}]\overleftrightarrow{R}(0) \right\} \quad (90)$$

where T is the time ordering operator, as before. The factor of $1/n!$ arises because the volume of an n -dimensional *simplex* (the n -dimensional generalization of the three dimensional tetrahedron), is $1/n!$ times the volume of the n -cube that contains it². Because we have chosen to integrate over the entire n -cube, we compensate by dividing out by $n!$ and introducing time-ordering.

¹Source: Wikipedia, <http://en.wikipedia.org/wiki/Tetrahedron>

²Source: Wikipedia, http://en.wikipedia.org/wiki/Simplex#Increasing_coordinates

Thus, the series expansion of \overleftrightarrow{R} can be written as

$$\begin{aligned}\overleftrightarrow{R}(t) &= \sum_{n=0}^{\infty} \overleftrightarrow{R}_n(t) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} T \left\{ \underbrace{\int_0^t dt_1 \dots \int_0^t dt_n}_{n \text{ integrations}} [\vec{\omega}^{body}(t_1) \cdot \overleftrightarrow{M}] \dots [\vec{\omega}^{body}(t_n) \cdot \overleftrightarrow{M}] \overleftrightarrow{R}(0) \right\}\end{aligned}\quad (91)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} T \left[\left(- \int_0^t dt' \left(\vec{\omega}^{body}(t') \cdot \overleftrightarrow{M} \right) \right)^n \right] \overleftrightarrow{R}(0) \quad (92)$$

where the exponent n implies that n such integrations are carried out iteratively. So,

$$\overleftrightarrow{R}(t) = \sum_{n=0}^{\infty} \frac{1}{n!} T \left[\left(- \int_0^t dt' \left(\vec{\omega}^{body}(t') \cdot \overleftrightarrow{M} \right) \right)^n \right] \overleftrightarrow{R}(0) \quad (93)$$

or equivalently,

$$\overleftrightarrow{R}(t) = T \left[\exp \left\{ - \int_0^t dt' \left(\vec{\omega}^{body}(t') \cdot \overleftrightarrow{M} \right) \right\} \right] \overleftrightarrow{R}(0) \quad (94)$$

Goldstein 5.8

The constants of the motion are:

$$E = \frac{1}{2} I_1 \omega_1^2 + \frac{1}{2} I_2 \omega_2^2 + \frac{1}{2} I_3 \omega_3^2 \quad (95)$$

$$L^2 = I_1^2 \omega_1^2 + I_2^2 \omega_2^2 + I_3^2 \omega_3^2 \quad (96)$$

We can treat these as equations involving ω_1 and ω_2 expressed as functions of ω_3 and the parameters E and L^2 , i.e. as a linear system of inhomogenous equations in ω_1^2 and ω_2^2 . The solutions are

$$\omega_1 = \pm \sqrt{\frac{(2E'I_2 - L')}{I_1(I_2 - I_1)}} \quad (97)$$

$$\omega_2 = \pm \sqrt{\frac{(2E'I_1 - L')}{I_2(I_1 - I_2)}} \quad (98)$$

where we have introduced the quantities E' and L' defined by

$$E' = E - \frac{1}{2} I_3 \omega_3^2 \quad (99)$$

$$L' = L^2 - I_3^2 \omega_3^2 \quad (100)$$

The signs of ω_1 and ω_2 cannot be known exactly with a knowledge only of E and L^2 (which are both quadratic in the velocities, and hence invariant under any permutation of the signs).

Now, Euler's equation for ω_3 is

$$I_3 \dot{\omega}_3 = \omega_1 \omega_2 (I_1 - I_2) \quad (101)$$

Substituting ω_1 and ω_2 from Eqns. (97) and (98) we get

$$I_3 \dot{\omega}_3 = \sqrt{\frac{(2E'I_1 - L')(L' - 2E'I_2)}{I_1 I_2}} \quad (102)$$

Substituting E' and L' from Eqns. (99) and (100), and rearranging, we get

$$\dot{\omega}_3 = \sqrt{\frac{[(2EI_1 - L^2) + I_3(I_3 - I_1)\omega_3^2][(L^2 - 2EI_2) + I_3(I_2 - I_3)\omega_3^2]}{I_1 I_2 I_3^2}} \quad (103)$$

Define the constants,

$$\alpha_1 = \frac{2EI_1 - L^2}{I_1 I_2 I_3^2}, \quad \alpha_2 = \frac{I_3(I_3 - I_1)}{I_1 I_2 I_3^2} \quad (104)$$

$$\beta_1 = \frac{L^2 - 2EI_2}{I_1 I_2 I_3^2}, \quad \beta_2 = \frac{I_3(I_2 - I_3)}{I_1 I_2 I_3^2} \quad (105)$$

In terms of these constants, Eqn. (103) can be written as

$$\dot{\omega}_3 = \sqrt{(\alpha_1 + \alpha_2 \omega_3^2)(\beta_1 + \beta_2 \omega_3^2)} \quad (106)$$

which when integrated, yields a mixed elliptic integral

$$t = \int_0^{\omega_3} \frac{du}{\sqrt{(\alpha_1 + \alpha_2 u^2)(\beta_1 + \beta_2 u^2)}} \quad (107)$$

taking the positive sign for $t > 0$. The elliptic integral can be rewritten as

$$t = \frac{1}{\sqrt{\alpha_1 \beta_1}} \int_0^{\omega_3} \frac{du}{\sqrt{(1 - k_1^2 u^2)(1 - k_2^2 u^2)}}$$

where

$$k_1^2 = -\frac{\alpha_2}{\alpha_1}, \quad k_2^2 = -\frac{\beta_2}{\beta_1} \quad (108)$$

With a change of variable ($v = k_2 u$), this can be written as

$$t = \frac{1}{\sqrt{-\alpha_1 \beta_2}} \int_0^{k_2 \omega_3} \frac{dv}{\sqrt{(1 - \xi^2 v^2)(1 - v^2)}}$$

that is,

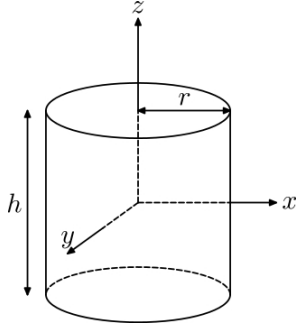
$$t = \frac{1}{\sqrt{(L^2 - 2EI_1)(I_2 - I_3)I_3}} \int_0^{\frac{\omega_3 I_3 (I_3 - I_2)}{(L^2 - 2EI_2)}} \frac{dv}{\sqrt{(1 - \xi^2 v^2)(1 - v^2)}} \quad (109)$$

where

$$\xi^2 = \frac{k_1^2}{k_2^2} = \frac{\alpha_2 \beta_1}{\alpha_1 \beta_2} = \frac{(L^2 - 2EI_1)(I_2 - I_3)}{(L^2 - 2EI_2)(I_1 - I_3)} \quad (110)$$

Once ω_3 has been determined (via an approximate solution to the elliptic integral), we can plug it back into Eqns. (97) and (98) to determine ω_1 and ω_2 .

Goldstein 5.14



A cylinder oriented along its three principal axes is shown in the figure³ on the left. Suppose its mass is m , radius is r and height is h . By symmetry, $I_x = I_y$.

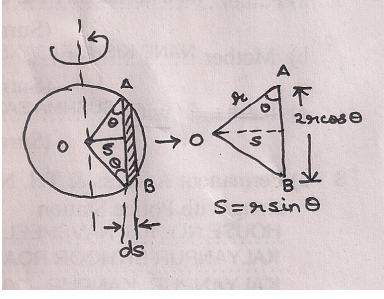
To compute I_z , we can regard the solid cylinder as made up of annular cylindrical shells from the center all the way up to r . Consider such a cylindrical shell of radius t (say), thickness dt , and height h . The moment of inertia of any such element about the z axis is $dI_z = t^2 dm$ where $dm = \rho \times (2\pi t dt h)$ is the mass of the shell, and $\rho = \frac{m}{\pi r^2 h}$ is the mass density of the solid cylinder. Therefore,

$$I_z = \int dm t^2 \quad (111)$$

$$\begin{aligned} &= \int \rho \times (2\pi t dt h) t^2 \\ &= \frac{2m}{r^2} \int_0^r dt t^3 \\ &= \frac{1}{2} m r^2 \end{aligned} \quad (112)$$

Now, to evaluate I_x , we can regard the solid cylinder as being made up of solid discs of radius r stacked vertically from $z = -h/2$ all the way to $z = +h/2$. Suppose any one such disc has a mass m' . We want to compute the moment of inertia of such a disc about an axis passing through its center of mass and lying in the plane of the disc.

³Figure taken from Wikipedia, http://en.wikipedia.org/wiki/File:Moment_of_inertia_solid_cylinder.svg.



Consider the cross section of the disc shown in the figure on the left. The disc can be regarded as a series of thin strips (shown shaded) located symmetrically about the rotation axis. Consider a strip at a distance s from the center O , having a width ds . As shown in the figure, $s = r \sin \theta$ and so $ds = r \cos \theta d\theta$. The angle θ ranges from $\pi/2$ at the right extreme to 0 at the center and finally $-\pi/2$ at the left extreme. The length of the strip is $2r \cos \theta$. Therefore, the mass element $dm = \frac{m'}{\pi r^2} \times (2r \cos \theta)(ds) = \frac{m'}{\pi r^2} (2r^2 \cos^2 \theta) d\theta$. So, the moment of inertia is

$$I_{||} = \int dm s^2 \quad (113)$$

$$= \int_{-\pi/2}^{\pi/2} \frac{m'}{\pi r^2} (2r^2 \cos^2 \theta) (r^2 \sin^2 \theta) d\theta$$

$$= \frac{1}{4} m' r^2 \quad (114)$$

Returning to the evaluation of I_x for the cylinder, we observe that dI_x , the contribution to the moment of inertia, due to rotation of the disc of mass dm' (situated at a point $(0, 0, z)$ on the rotation axis \hat{x}) is

$$dI_x = \frac{1}{4} dm' r^2 + dm' z^2 \quad (115)$$

using the parallel axis theorem. Also, $dm' = \frac{m}{\pi r^2 h} \times (\pi r^2 dz) = \frac{m}{h} dz$ and hence,

$$I_x = \int dI_x \quad (116)$$

$$= \frac{m}{h} \int_{-h/2}^{h/2} dz \left(\frac{r^2}{4} + z^2 \right)$$

$$= \frac{1}{4} m r^2 + \frac{1}{12} m h^2 \quad (117)$$

So, $I_x = I_y = \frac{1}{4} m r^2 + \frac{1}{12} m h^2$ and $I_z = \frac{1}{2} m r^2$. The condition for the ellipsoid at the center of the cylinder to be a sphere is $I_x = I_y = I_z$, i.e.

$$\boxed{\frac{1}{4} m r^2 + \frac{1}{12} m h^2 = \frac{1}{2} m r^2} \quad (118)$$

which gives

$$\frac{\text{height}}{\text{diameter}} = \frac{h}{2r} = \frac{\sqrt{3}}{2} \quad (119)$$