

Relating scattering angle of particle ① in laboratory ^(lab) frame (θ_{lab}) to that in center-of-mass (COM) frame (θ_{com}) [based on GPS, sec. 3.11]

[Recall that $\theta_{com} = \theta$ of effective 1-particle scattering, i.e., with fixed center of force, as in GPS sec. 3.10 and previous lectures. Thus above relation will connect also θ_{lab} to θ so that we can transfer analysis done with θ , e.g., computation of cross-section (σ) to lab frame.]

Outline of procedure

- Basic idea is to use "velocity triangle" (i.e., relate velocities of particle 1 in COM & lab frames): after all, θ 's are also directions of velocities
 - first, obtain relation between $\cos \theta_{com}$ and $\cos \theta_{lab}$ in terms of v_0/v_1' , where v_0 is initial speed of particle 1 in lab frame and v_1' is its final speed in COM frame
 - then use v_0/v , where v is final relative speed of 2 particles
 - why above step? Because v_0/v can be conveniently expressed in terms of (kinetic) energy loss factor

called "Q-value" of process (where elastic collision, i.e., no kinetic energy lost, corresponds to $Q=0$) (2)

[see Eq. (6)] on page 3

— Use above $\theta_{\text{com}} (= \theta) - \theta_{\text{lab}}$ relation to convert $\sigma(\theta)$ [obtained earlier] into $\sigma'(\theta_{\text{lab}})$, i.e., what's actually measured: [see Eq. (12)] on page 5

— simple, specific cases / checks

— consider energy transferred by particle 1 to particle 2 (latter being at rest initially) in lab frame

Onto details!

— Notation: $[\vec{r}_1, \vec{v}_1]$ for particle (1) after scattering in lab frame

$[\vec{r}'_1, \vec{v}'_1]$ for particle (1) after scattering in COM frame

$[\vec{R}, \vec{V}]$ for (COM) in lab frame (of course, \vec{V} is constant) particle 2 in lab frame

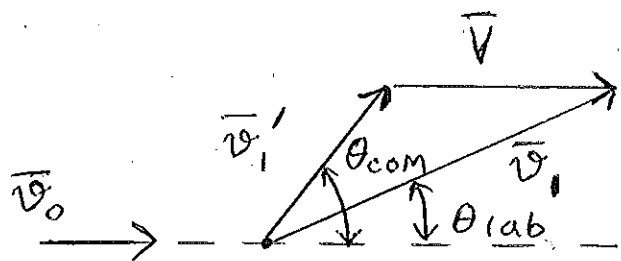
— Using $\vec{R} = (\vec{r}_1 m_1 + \vec{r}_2 m_2) / (m_1 + m_2)$ and $\vec{r} = \vec{r}_2 - \vec{r}_1$, we have

$$\vec{r}'_1 = -m_2 \vec{r} / (m_1 + m_2) \text{ so that } \vec{v}'_1 = (\mu / m_1) \vec{v} \dots (1)$$

where \vec{v} is relative velocity (\vec{r} is relative position vector) after collision and $\mu = m_1 m_2 / (m_1 + m_2)$ is the reduced mass.

— Basic Velocity triangle: clearly $\vec{r}_1 = \vec{R} + \vec{r}'_1$ so that $\vec{v}_1 = \vec{V} + \vec{v}'_1 \dots (2)$

or in picture form:



— Relating \bar{V} to \bar{v}_0 : assuming particle 2 is at rest initially in lab frame, conservation of (linear) momentum gives in lab' frame

$$\begin{aligned}
 m_1 \bar{v}_0 \text{ (initial)} &= m_1 \bar{v}_1 + m_2 \bar{v}_2 \text{ (final)} \quad \begin{array}{l} \nearrow \text{2nd particle} \\ \text{in COM} \\ \text{frame} \end{array} \\
 &= m_1 (\bar{V} + \bar{v}_1') + m_2 (\bar{V} + \bar{v}_2') \\
 &= (m_1 + m_2) \bar{V} + \underbrace{(m_1 \bar{v}_1' + m_2 \bar{v}_2')}_{\text{total momentum (after collision) in COM frame} = 0}
 \end{aligned}$$

i.e., $\bar{V} = m_1 \bar{v}_0 / (m_1 + m_2)$
 $= (\mu / m_2) \bar{v}_0 \quad \dots (3)$

Obviously, \bar{V} is in same direction as \bar{v}_0 , as already used in above picture] (by definition of COM)

— Back to velocity triangle : matching horizontal components of Eq. (2) [or picture] gives

$$v_1 \cos \theta_{lab} = \frac{v_1'}{er} \cos \theta_{COM} + V \quad \dots (4)$$

whereas considering lengths of sides of triangle gives

$$v_1^2 = (v_1' \cos \theta_{COM} + V)^2 + (v_1 \sin \theta_{COM})^2$$

$$= v_1'^2 + V^2 + 2 v_1' V \cos \theta_{COM} \quad \dots (5)$$

— Using magnitude of Eq. (3) in Eqs. (4), (5) gives 1st relation between 2 θ 's :

$$\cos \theta_{lab} = (\cos \theta_{COM} + \rho) / \sqrt{1 + \rho^2 + 2\rho \cos \theta_{COM}} \quad \dots (6)$$

where $\rho \equiv \frac{\mu}{m_2} \frac{v_0}{v_1} \dots (7)$

- However, v_1' ^{→ speed after collision in COM frame} appearing in ratio of speeds in Eq. (7) is not yet known (while v_0 is)

- So, we trade v_1' for v using Eq. (1) to give

$\rho = (m_1/m_2) (v_0/v) \dots (8)$

- Well, v (speed ^{relative} after collision) is also not yet known, but can be obtained using conservation

of energy: in effective 1-body picture, (assuming original collision is elastic) it is clear that speed of particle is same ^(well) before and after collision (since potential energy = 0 in both these positions so that we can set kinetic energies

to be equal), i.e.,
relative speed before collision

$v_0 = v$ (elastic) ... (9) (used in above argument)

[Recall effective 1-particle velocity/position is really the relative velocity/position of (2) particles, in any inertial frame, ^{which is how v, v_0 "appeared" initially}] same (total)

- In general, i.e., if kinetic energy is lost in collision, we define "Q-value" (energy-loss factor) of collision by

$\mu v_2^2 = \mu v_0^2 / 2 + Q \dots (10)$ so that using Eq. (10)
in Eq. (8)
+ algebra

final KE of effective 1-body

initial...

$\rho = (m_1/m_2) / \sqrt{1 + \frac{m_1+m_2}{m_2} \frac{Q}{E}} \dots (11)$

where $E = \frac{1}{2} m v_0^2$ is ^{initial} energy of particle 1 in lab frame

Onto cross-sections

(5)

— Connecting $\sigma[\theta_{\text{com}} (= \theta)]$ to $\sigma'(\theta_{\text{lab}})$ involves a sort of Jacobian of $\theta_{\text{com}} \rightarrow \theta_{\text{lab}}$ "transformation", i.e., equating number of particles in ^{given} solid angle element is same, whether expressed in terms of θ_{lab} or θ :

$$2\pi I \sigma(\theta) \sin \theta d\theta = 2\pi I \sigma'(\theta_{\text{lab}}) \sin \theta_{\text{lab}} d\theta_{\text{lab}}$$

so that using Eq. (6) (+ algebra) gives

$$\sigma'(\theta_{\text{lab}}) = \sigma(\theta) (1 + \rho^2 + 2\rho \cos \theta)^{3/2} / (1 + \rho \cos \theta) \dots (12)$$

[Again, both σ 's are measured in lab frame, but just written in terms of different θ 's.]

— Onto specific cases

(a) elastic collision and $m_1 = m_2$ so that $\rho = 1$
[as per Eqs. (8), (9)], giving [using Eq. (6)]

$$\cos \theta_{\text{lab}} = \sqrt{\frac{1 + \cos \theta}{2}} = \cos \frac{\theta}{2} \Rightarrow \boxed{\theta_{\text{lab}} = \theta/2}$$

i.e., scattering angle in lab frame $\leq \pi/2$ (since $\theta \leq \pi$)
[only ⁱⁿ forward hemisphere]

and $\sigma'(\theta_{\text{lab}}) = 4 \cos \theta_{\text{lab}} \sigma(\theta)$ so that isotropic scattering in θ does not imply same in lab frame

(b) $m_2 \gg m_1$ and elastic gives $\rho \approx 0$, i.e.

$\theta_{\text{lab}} = \theta$ as expected: in lab frame, recoil of ^(at rest initially) 2nd (very heavy) particle is then negligible so that lab frame \approx effective 1-body picture.

→ Finally, even for elastic collision (again, (6) total KE is constant), (1st) particle transfers part of its initial energy to 2nd (target) particle: this degree of slowing down is

given by $\frac{v_1^2}{v_0^2}$ \leftarrow final speed in lab frame

$$\frac{v_1^2}{v_0^2} \leftarrow \text{initial...} = \left(\frac{\mu}{m_2 \rho} \right)^2 (1 + \rho^2 + 2\rho \cos \theta) \quad \dots (13) \quad \left(\begin{array}{l} \text{in} \\ \text{general} \end{array} \right)$$

by plugging v_1' in terms of ρ, v_0 [from Eq. (7)] and V in terms of v_0 [from Eq. (3)] into RHS of Eq. (5)

from Eqs. (8), (9)

→ for elastic collision ($\rho = m_1/m_2$), we get

$$\frac{E_1}{E_0} = \frac{(1 + 2\rho \cos \theta + \rho^2)}{(1 + \rho)^2} \quad \text{[elastic]}$$

(ratio of energies of particle 1 before/after collision)

use $\theta_{\text{lab}} = \theta/2$

$$\rightarrow \frac{(1 + \cos \theta)}{2} = \cos^2 \theta_{\text{lab}} \quad \text{for } \boxed{m_1 = m_2}$$

i.e., for maximum scattering angle ($\theta_{\text{lab}} = \pi/2$), all of energy of incident particle is transferred to 2nd (of same mass), i.e., incident particle is brought to rest