$\boxed{\text{Phys 601}}$ by Dr. Agashe

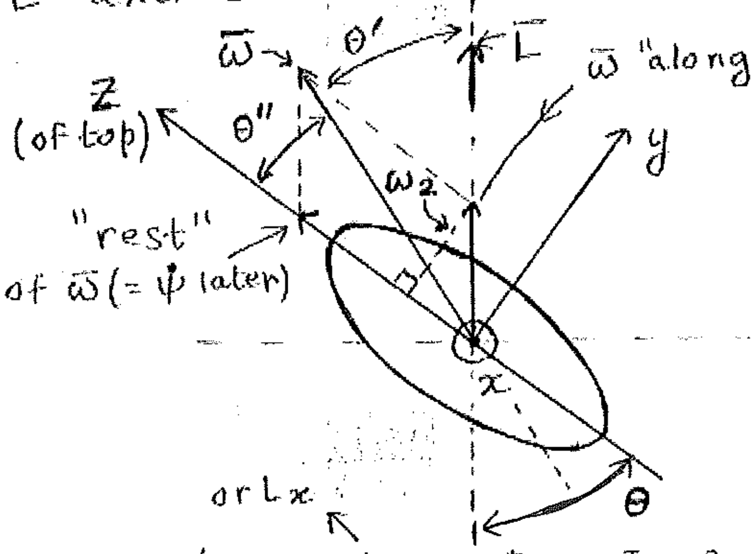
$\boxed{\text{Notes on free, symmetrical top}}$ ($I_1 = I_2$ (or I_{xx} or I_{yy}) $\neq I_3$ (or I_{zz})) : 3 different approaches giving same results (based on Landau, Lifshitz (LL) § 33, 35 & 36)

$\boxed{\text{(I)}}$. Using law of conservation of angular momentum

— $\boxed{\bar{L}}$ = constant : choose it to be vertical

— $\boxed{z}$ (symmetry) axis of top (henceforth, often called simply "axis" of top) and $\bar{L}$ in plane of paper, but at an angle θ relative to each other

— Using symmetry about top axis, $\boxed{\text{choose}}$ (any) principal axes $\boxed{x \,\&\, y}$ so that, at given instant, $\boxed{x}$-axis is perpendicular to plane of $\bar{L}$ and z-axis.



("similar" to Fig. 46 on page 107 of L.L.)

— Thus, we have $L_1 = 0 = I_1 \omega_1 \Rightarrow \omega_1 = 0$ so that $\boxed{\bar{\omega}}$ is in same plane as $\bar{L}$ and z-axis of top (again, latter axis is that of "usual" rotation or "spin" of top) upper(lower) half

— In turn, velocity of any point on z-axis ②
of top (given by $\bar{\omega} \times \bar{r}$ as usual) is
out of (into) this plane, (and this is valid at any instant) i.e., axis of top rotates
about direction of $\bar{L}$ (called precession),
again in addition to top itself rotating
about its own (z-) axis $\Leftrightarrow$ $\boxed{\bar{\omega}}$ also rotates about $\boxed{\bar{L}}$ with ω_{pr} (since $\bar{\omega}$, $\bar{L}$, z-axis in 1 plane)

— Onto formulae: had $\dot{\theta}$ been non-zero,
a point on axis of top would have a velocity
component in plane of figure. However,
we argued above (based on $\omega_1 = 0$ or $\bar{\omega}$ being
in plane) that velocity of axis is purely out
of plane $\Rightarrow$ $\dot{\theta}$ must vanish, i.e., $\boxed{\theta = \text{constant}}$ ...(1)

— $\boxed{\text{Spin}}$ of top is just component of $\bar{\omega}$ along
its (z-) axis, i.e., $\boxed{\omega_3} = L_3/I_3 = \dfrac{L\cos\theta}{I_3 \ (= \text{constant})}$ ...(2)

— In order to determine rate of precession
(dentoted by ω_{pr}), we resolve $\bar{\omega}$ into
components along $\bar{L}$ and z-axis: see figure
(where latter is denoted by "rest" of $\bar{\omega}$: note
that this is not quite ω_3 of above, since the former
component, i.e, along $\bar{L}$, is not $\perp$ to z-axis!)

[For later discussion, we point out that these
2 components are $\dot{\phi}$, i.e, rotation about
space z-axis ($\bar{\bar{e}}_3$), and $\dot{\psi}$, i.e, rotation about $\bar{e}_3$:
$\bar{\omega} = \dot{\phi}\,\bar{\bar{e}}_3 + \dot{\psi}\,\bar{e}_3 + \underbrace{\dot{\theta}}_{\text{vanishes here}}\,\underbrace{\bar{e}_1'}_{\text{line of nodes}}$]

— The latter/above "component" of $\overline{\omega}$ along z-axis does not give any displacement of z-axis of top $\Rightarrow$ former component along $\overline{L}$ must be the ω_{pr} needed

— Now, from figure, we see that

$\omega_{pr} \times \sin\theta = \omega_2$ since rest of $\overline{\omega}$ (along z-axis has no component along y-axis). But,

$$\omega_2 = L_2/I_1 = L\sin\theta/I_1 \Rightarrow \boxed{\omega_{pr.}} = \frac{L}{I_1} \; (= \text{constant}) \quad \dots (3)$$

— × —

— We can be $\boxed{\text{ambitious}}$ by deducing additional results (as follows)! Clearly $|\overline{\omega}|^2 = \omega_3^2 + \omega_1^2$

$$= L^2\left(\cos^2\theta/I_3^2 + \sin^2\theta/I_1^2\right) = \text{constant}.$$

— Combining this with ω_3 being constant shows that $\overline{\omega}$ is at fixed angle relative to $\boxed{z}$-axis (θ'' in figure). Since angle between z-axis and $\overline{L}$ was already shown to be constant, so is that between $\boxed{\overline{\omega}}$ and $\boxed{\overline{L}}$ (θ' in figure), i.e., $\theta' + \theta'' = \theta$ (it's crucial to use here $\overline{\omega}$, $\overline{L}$ and z-axis being in same plane)

— We can relate $\boxed{\theta'}$ to $\boxed{\theta''}$ (thus obtaining both in terms of θ) as follows. Return to decomposition of $\overline{\omega}$ into ω_{pr} (along L) $[= L/I_1]$ and along top axis; latter is given by $\left[\omega_3 - L\cos\theta/I_1\right]$, i.e., we should get spin of top (ω_3) when the latter component is $\boxed{\text{added}}$ to component of ω_{pr} $(= L/I_1)$ along z-axis [again, for later discussion, this latter component is $\dot{\psi} = \omega_3 - \dot{\phi}\cos\theta$]

— Now, we must have

$$\underbrace{L/I_1}_{\substack{\omega_{pr} \\ \bar{\omega} \text{ along } \bar{L}}} \times \underbrace{\sin\theta'}_{\substack{\text{to give} \\ \text{component } \perp \\ \text{to } \bar{\omega}}} = \underbrace{\left(\omega_3 - \cos\theta \frac{L}{I_1}\right)}_{\substack{\text{"rest" of } \bar{\omega} \\ \text{(along z-axis)}}} \times \underbrace{\sin\theta''}_{\substack{\text{to give} \\ \text{component} \\ \perp \text{ to } \bar{\omega}}}$$

i.e., net component of $\bar{\omega}$ $\perp$ to $\bar{\omega}$ $= 0$ (!) $\Rightarrow$

$$\boxed{\sin\theta'/\sin\theta''} = \frac{\left(\omega_3 - \frac{L}{I_1}\cos\theta\right)}{L/I_1} = \frac{L\cos\theta\left(\frac{1}{I_3} - \frac{1}{I_1}\right)}{L/I_1}$$

use Eq.(2)

$$= \cos\theta\left(I_1/I_3 - 1\right) \quad \dots (4)$$

— x —

$\boxed{\text{(II)}}$. Next, connect (or match) to $\boxed{\text{Euler angles}}$

- choose $\bar{L}$ to be along <u>space</u> $\bar{z}$-axis ($\bar{\tilde{e}}_3$) so that polar angle of <u>top axis</u> ($\bar{e}_3$) w.r.t. $\bar{\tilde{e}}_3$ is θ,
- whereas its <u>azimuthal angle</u> is $(\phi - \pi/2)$: this can be "seen" from figure below, i.e. by dropping perpendicular from tip of $\bar{e}_3$ axis onto $\bar{\tilde{e}}_1$-$\bar{\tilde{e}}_2$ (i.e., space x-y) plane or explicitly, use $\bar{\tilde{e}} = R^{-1}\bar{e} = R^T\bar{e}$ with $\mathbf{e} = (0\ 0\ 1)^T$, i.e., find $\bar{e}_3$ components along space axes:

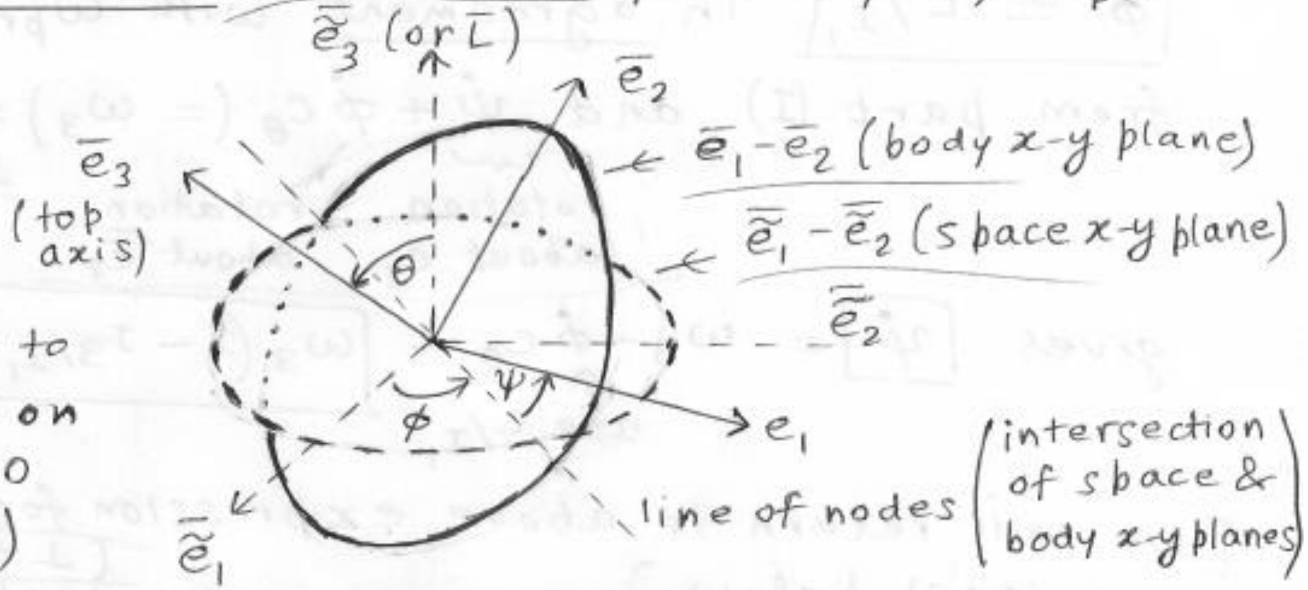
$$\bar{e}_3 = s_\theta s_\phi \bar{\tilde{e}}_1 - s_\theta c_\phi \bar{\tilde{e}}_2 + c_\theta \bar{\tilde{e}}_3$$

$$= s_\theta \cos(\phi - \pi/2)\, \bar{\tilde{e}}_1 + s_\theta \sin(\phi - \pi/2)\bar{\tilde{e}}_2 + c_\theta \bar{\tilde{e}}_3$$

cos/sin of azimuth

— Also, note that $\boxed{\text{line of nodes}}$ (called earlier $\bar{e}_1'$) $\perp$ <u>both</u> $\bar{e}_3$ & $\bar{\tilde{e}}_3$

— Thus, $\boxed{\dot{\phi}}$ corresponds to angular velocity of precession discussed in part I, i.e., ω_{pr}



(similar to Fig. 47 on page 110 of LL)

— Onto formulae: we have (see HW 6.5) again

ω_1 ($\bar{\omega}$ component along body x-axis) $= \dot{\phi} s_\theta s_\psi + \dot{\theta} c_\psi$;

$\omega_2 = \dot{\phi} s_\theta c_\psi - \dot{\theta} s_\psi$ and $\omega_3 = \dot{\phi} c_\theta + \dot{\psi}$

— As before, we can use symmetry about z-axis of top ($\bar{e}_3$) in order to choose at $\boxed{\text{given}}$ instant (see later for what happens at "next" instant!) x-y axes such that $\boxed{\psi = 0}$ or $\boxed{\bar{e}_1}$ is line of $\boxed{\text{nodes}}$ so that

$\omega_1 = \dot{\theta}$; $\omega_2 = \dot{\phi} s_\theta$ (while $\omega_3 = \dot{\psi} + \dot{\phi} c_\theta$) ... (5)

— On the other hand, we have (with $\psi = 0$)

$L_1 = 0$ (since $\bar{e}_1$ chosen to be $\perp$ to $\tilde{\bar{e}}_3$, i.e., $\bar{L}$);

$L_2 = L \sin\theta$ and $L_3 = L\cos\theta$ ... (6)

— Matching (5) & (6) [using $L_a = I_a \omega_a$ (no sum over a!)]

we get $\dot{\theta}\ (= \omega_1) = L_1 / I_1 = 0$, i.e., $\boxed{\theta = \text{constant}}$;

[as in Eq. (1) of part (I)]

$\dot{\phi}\, s_\theta \; (= \omega_2) = L_2/I_1 = L \sin\theta / I_1 \Rightarrow$ (6)

$\boxed{\dot{\phi} = L/I_1}$ in <u>agreement</u> with ω_{pr} in Eq. (3)

from part (I) and $\underbrace{\dot{\psi}}_{\substack{\text{rotation} \\ \text{about } \bar{e}_3}} + \underbrace{\dot{\phi}\, c_\theta}_{\substack{\text{rotation} \\ \text{about } \bar{\bar{e}}_3}} (= \omega_3) = L_3/I_3 = \dfrac{L c_\theta}{I_3}$

gives $\boxed{\dot{\psi}} = \omega_3 - \underset{\substack{\uparrow \\ \text{use } L/I_1}}{\dot{\phi}}\, c_\theta = \boxed{\omega_3 \left(1 - I_3/I_1\right)} \ldots (7)$

[we will return to above expression for $\dot{\psi}$ in part (III) below.]

— It is also clear that $\boxed{\bar{\omega}, \bar{L} \text{ \& top axis}}$ are all in <u>one plane</u> ($\perp$ to $\bar{e}_1$ / line of nodes)

[Of course, we also (re-)obtain $|\bar{\omega}| = \sqrt{\omega_3^2 + \omega_2^2}$ = constant etc. in this way.] (again, body axes components)

— Now, at $\boxed{\text{next}}$ instant, say after time δt, we of course get a non-zero (even if small) ψ, i.e., $\boxed{\delta\psi = \dot{\psi}\, \delta t}$ ($\psi > 0$ is "anticlockwise")

— However, using the same "freedom" in choosing body (principal) x-y axes, we can (again) "re-define" (slightly) body x-y axis so as to get back earlier picture: concretely, we choose <u>new</u> x-axis (call it $\bar{e}_1^{(new)}$) to be rotated w.r.t. old x-axis $\bar{e}_1$ by $(-\delta\psi)$, i.e., $\delta\psi$ in clockwise direction.

[this is just rotation of body x-y plane <u>about</u> $\bar{e}_3$]

In this way, $\bar{e}_1^{(new)}$ is still along line of nodes so that $\psi^{(new)} = 0 \Rightarrow L_1^{new} = 0$ etc., i.e., $\boxed{\omega_1^{(new)} = 0}$ (as before)

— In other words, $\boxed{\text{if}}$ we keep "changing" (7)
body x-y axes (as body itself is moving in space frame) in above fashion [i.e, with "angular velocity" of $\theta \dot{\psi}$ about top axis $(\bar{e}_3)$] then $\bar{\omega}$ seems to be "fixed" in body frame [i.e., $\omega_3 = |\bar{\omega}| \cos\theta$; $\omega_2^{\text{new}} = |\bar{\omega}| \sin\theta$ & $\omega_1^{\text{new}} = 0$]!

— However, above rotation/re-definition of body x-y axes was merely a "trick" (e.g., designed to show $\dot{\theta} = 0$, compute precession etc.): in reality, we work with fixed body x-y axes (like "new" body axes) $\Rightarrow$ $\boxed{\bar{\omega}}$ is actually rotating with angular velocity $\boxed{-\dot{\psi}}$ about axis of top (again, this is body-frame viewpoint): we will return to this point just below.

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— Again, let's do more checks (as follows): resolve $\bar{\omega}$ along space-frame axes (cf. body frame done earlier): from HW 5.5, we get [for our case of $\dot{\theta} = 0$] $\tilde{\omega}_1$ (again, $\bar{\omega}$ component along space x-axis) $= \dot{\psi}\, s_\theta\, s_\phi$; $\tilde{\omega}_2 = -\dot{\psi}\, s_\theta\, c_\phi$ and $\tilde{\omega}_3 = \dot{\psi}\, c_\theta + \dot{\phi}$.

— So, we find (i) $|\bar{\omega}| = \sqrt{\tilde{\omega}_1^2 + \tilde{\omega}_2^2 + \tilde{\omega}_3^2} = \sqrt{\dot{\phi}^2 + \dot{\psi}^2 + 2c_\theta \dot{\psi}\dot{\phi}}$, i.e, constant and (ii) azimuthal angle of $\bar{\omega}$ is $(\phi - \pi/2)$ [since $\tilde{\omega}_2/\tilde{\omega}_1 = \tan(\phi - \pi/2)$], i.e, $\boxed{\bar{\omega}}$ rotates about space z axis at rate of $\dot{\phi}$

(III). Finally, we use Euler's $\boxed{\text{equations}}$, i.e., for time-dependence of components of $\bar{\omega}$ along body axes. We have (with $\boxed{I_1 = I_2}$)

$I_3 \dot{\omega}_3 = 0 \Rightarrow \boxed{\omega_3 = \text{constant}}$ (as before)
(i.e., spin/rotation about top axis)

$\Rightarrow \boxed{\cos\theta} = L_3 \, (= I_3 \omega_3)/|L| \boxed{= \text{constant}}$

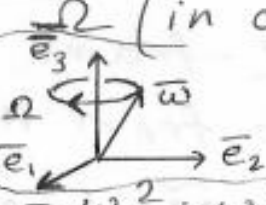
$|L|$ constant (again, as before)

[Note that $I_1 = I_2$ is crucial here: for $I_1 \neq I_2$ (asymmetric top), we have (in general) $I \dot{\omega}_3 = -\omega_1 \omega_2 (I_2 - I_1) \neq 0$, i.e., $\omega_3 \neq$ constant]

— However, $\dot{\omega}_{1,2} \neq 0$, i.e., $I_1 \dot{\omega}_1 = + \omega_2 \Omega I_1$,
where $\boxed{\Omega}$ (as in DT's notes) $\equiv \omega_3 (I_1 - I_3)/I_1$ $\quad \cdots (8)$
note GPS, LL have opposite sign here; constant

and $\dot{\omega}_2 = -\omega_1 \Omega \Rightarrow \ddot{\omega}_1 = -\Omega^2 \omega_1$

— Solution is $\omega_{1,2} = \overbrace{\omega_0}^{\text{constant}} (\sin \Omega t, \cos \Omega t)$,
i.e., component of $\boxed{\bar{\omega}}$ $\perp$ to axis of top (itself rotating in space frame) $[= \omega_1 \bar{e}_1 + \omega_2 \bar{e}_2]$ rotates about top axis with angular velocity Ω (in clockwise direction for $\Omega > 0$): (Fig. 30 on page 56 of DT)



$\Rightarrow \boxed{|\bar{\omega}|^2} = \omega_3^2 + \omega_1^2 + \omega_2^2 = \omega_3^2 + \omega_0^2 \boxed{= \text{constant}}$

— Since ω_3 (= component along $\bar{e}_3$) is also constant, it is clear that entire $\bar{\omega}$ rotates about axis of top at rate Ω

— Since $L_{1,2} = \underbrace{I_{1,2}}_{\substack{\text{same} \\ \text{(constant)}}} \underbrace{\omega_{1,2}}_{\substack{\text{different} \\ \text{(changing with time)}}}$ and $L_3 = I_3 \omega_3 = \text{constant}$, we also see that (entire) $\bar{L}$ rotates about axis of top with angular velocity Ω (9)

[again, this is motion of $\bar{L}$ – as "defined/constructed" by space/inertial frame observer, i.e., $\sum_i m_i \bar{r}_i \times \frac{d\bar{r}_i}{dt}$ as observed in space frame – but as resolved along body axes (or "seen" by body observer, i.e., $\bar{L} = \sum_a L_a (\bar{e}_a)$, constant for body observer
$L_3 = \text{constant}$, but $\dot{L}_{1,2} = I_{1,2}\dot{\omega}_{1,2} \neq 0$]

— x —

— Match the above rotation of $\boxed{\bar{\omega}}$ (or $\bar{L}$) about top axis as obtained from Euler's equations in part (III) just above to what we earlier concluded in part (II) using Euler angles, i.e., $\boxed{\Omega}$ "should" equal $\boxed{\dot{\psi}}$ (both clockwise) : indeed Eq.(7) for $\dot{\psi}$ is same as Eq.(8) for Ω

— Equivalently (more explicitly), we can look at rotation of $\boxed{\bar{L}}$ in body frame, which is along $\tilde{\bar{e}}_3$. So, expand $\tilde{\bar{e}}_3$ in $\bar{e}_{1,2,3}$ [using $\bar{e} = R\tilde{\bar{e}}$ and setting $\tilde{\bar{e}} = (0,0,1)^T$] giving

$$\tilde{\bar{e}}_3 = \underbrace{S_\theta S_\psi \bar{e}_1 + S_\theta C_\psi \bar{e}_2}_{\text{constant}} + C_\theta \bar{e}_3$$

shows clockwise rotation about $\bar{e}_3$ at rate $\dot{\psi}$ about $\bar{e}_3$

i.e., $\bar{L}$ rotates (clockwise) about top axis at rate $\dot{\psi}$, (again) matching Ω obtained in part (III).

— Finally, we can come a "full circle" by seeing from way (I) the above rotation of $\bar{\omega}$ (or $\bar{L}$) about top axis (i.e., from $\boxed{\text{space}}$ frame/viewpoint)

— It's $\boxed{\text{easier}}$ to consider $\boxed{\bar{L}}$, which is fixed in space-frame (cf. $\boxed{\bar{\omega}}$ moving in space-frame also!).

— Begin with ω_{prec} component of $\bar{\omega}$ (i.e., along $\bar{L}$/vertical space axis): it gives an instantaneous velocity to entire body x-axis (coming out of z-$\bar{L}$ plane) along horizontal/right in Figure on page 1.

$\Rightarrow$ body x-axis remains $\perp$ to $\bar{L}$ in this process, i.e., L_1 stays 0 (also, $L_2 = L \sin\theta$ and $L_3 = L\cos\theta$ even for new y', z' axis): clearly $\bar{L}$ does $\boxed{\text{not}}$ move even in body-frame (again, on account of ω_{prec} only)

— There remains "rest" of $\bar{\omega}$ component (see figure on page 1) along top-axis: as discussed before, $\boxed{\text{this}}$ does not give any displacement of top (since this $\bar{\omega} \parallel z$-axis). It corresponds to simply rotation of body x-y plane (in space frame of course!) about z-axis, while $\bar{L}$ is fixed

— Thus, for an observer $\boxed{\text{fixed}}$ in body, it is $\bar{L}$ which is rotating at rate of $\ominus$ "rest" of $\bar{\omega}$.

— Now, "rest" of $\bar{\omega}$ (see bottom of page 3/top of page 4)

$$= \omega_3 - L\cos\theta/I_1 = \omega_3 - L_3/I_1 = \omega_3 - I_3\omega_3/I_1 = \omega_3\left(1 - \frac{I_3}{I_1}\right)$$

so that $\bar{L}$ is rotating in body frame about z-axis "clockwise" at rate of $\omega_3(1 - I_3/I_1)$, matching $\dot{\psi}$ of (II) and Ω of (III)