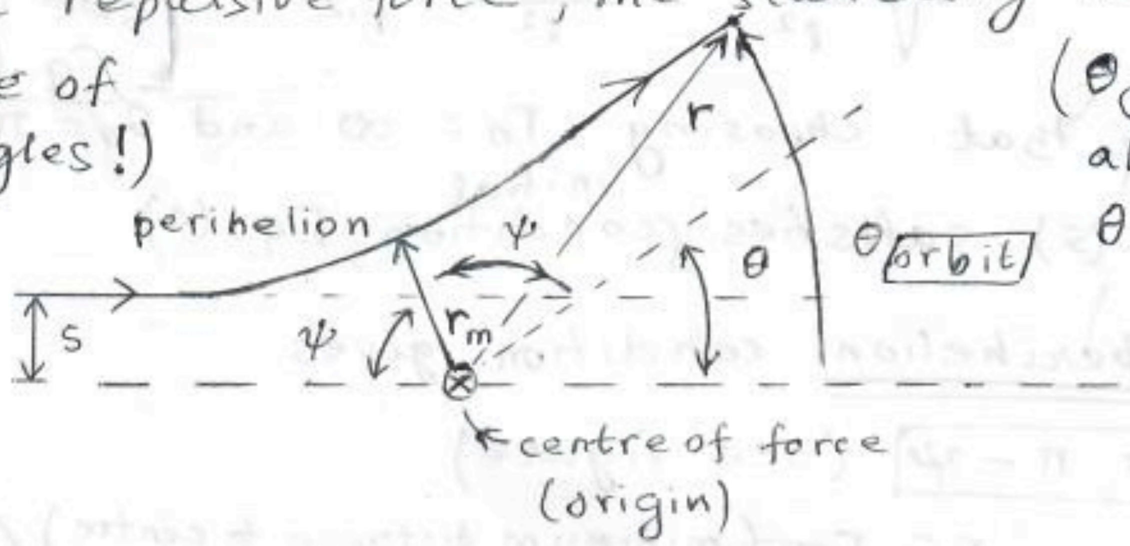
Phys 601 (Fall, 2018) by Dr. Agashe (1)

$\boxed{\text{Relating}}$ impact parameter(s) to scattering angle (θ) for equivalent 1-body scattering by central force field (based on GPS sec. 3·10)

For repulsive force, the scattering looks like

(A tale of (3) angles!)



(θ_{orbit} varies along path: θ is fixed)

At perihelion, angle between position vector and initial direction of particle (see figure) is $\boxed{\psi}$ so that

$$\theta = \pi - 2\psi \quad \cdots (1)$$

(use symmetry of orbit about perihelion)

and distance from centre of force is r_m, which (as per equivalent 1-D potential analysis) is given by

$$\underbrace{E}_{\text{constant}} = V'(r_m) = V(r_m) + \frac{l^2}{2 m r_m^2} \quad \cdots (2)$$

with l being "traded" for s using

$$\underbrace{l}_{\text{constant}} = m \underbrace{v_0}_{\text{initial/final speed}} s \quad \text{and} \quad E = \tfrac{1}{2} m v_0^2 \Rightarrow l = s\sqrt{2mE} \quad \cdots (3)$$

Also, choose convention for θ_{orbit} as shown, i.e., initial position $(r=\infty)$ corresponds to $\theta = \pi$ ... (4)

Since the general expression for $\theta_{orbit}(r)$ is

$$\theta_{orbit} = \int_{r_0}^{r} \frac{dr'/r'^2}{\sqrt{\frac{2mE}{\ell^2} - \frac{2mV(r')}{\ell^2} - \frac{1}{r'^2}}} + \theta_0 \quad \cdots (5)$$

[see GPS, Eq. (3.36)]

we see that choosing $r_0 = \infty$ and $\theta_0 = \pi$ in Eq. (5) satisfies initial condition, Eq. (4)

Then, perihelion condition gives

$$\boxed{\theta_{orbit} = \pi - \psi} \text{ (see figure)}$$

$$= \int_{\infty}^{r = r_m \text{ (minimum distance to centre)}} \frac{dr'/r'^2}{\sqrt{2mE/\ell^2 - 2mV(r')/\ell^2 - 1/r'^2}} + \pi \quad \text{[use Eq (4)]}$$

θ_0

i.e., $\psi = \int_{r_m}^{\infty} \frac{dr'/r'^2}{\sqrt{2mE/\ell^2 - 2mV(r')/\ell^2 - 1/r'^2}}$, giving

[from Eq. (1) and using last of Eq. (3)]

$$\boxed{\theta(s) = \pi - 2\int_{r_m}^{\infty} \frac{dr\,(s)}{r\sqrt{r^2\left[1 - \frac{V(r)}{E}\right] - s^2}}} \quad \cdots (6)$$

(scattering angle)

with $\boxed{r_m}$ obtained in terms of E and s by solving Eq. (2), plugging in last of Eq. (3)

— A $\boxed{\text{check}}$: from figure, it is clear that final position corresponds to $\boxed{\theta_{orbit} = \theta}$, with $r \to \infty$ (like initial position). But, then plugging $r \to \infty$ into Eq. (5) — $\boxed{\text{naively}}$ — gives $\theta_{orbit} = \pi$ (since we chose $r_0 = \infty$ and $\theta_0 = \pi$ to satisfy initial condition), i.e., (obviously) same as initial position!

— Resolution: we need to back-up from Eq. (5): rewrite is as (schematically) ← drop "orbit" for ease (see previous notes)

$$\theta_{orbit} = \int^r \frac{dr' \quad \ell/(m r'^2) \Big\} d\theta/dt}{\sqrt{2/m\left[E - V(r') - \frac{\ell^2}{2mr'^2}\right]} \Big\} dr/dt}$$

$$= \int dr' / (dr'/d\theta)$$, which is how we again got Eq. (5) in first place

— Now, r and θ_{orbit} are $\boxed{\text{both}}$ decreasing as we go from initial to perihelion position (see figure) so that $dr/d\theta > 0$ along this part of trajectory. Thus, taking positive sign of square root in Eq. (5) was indeed (a posteriori!) justified: recall that, at that stage, we only used Eq. (5) in this regime.

— However, from perihelion to final position (i.e., $r = r_m$ to $r = \infty$), $dr/d\theta < 0$ (i.e., r is increasing, while θ_{orbit} continues to reduce) so that we should take negative sign for square root, i.e., integral in Eq. (5) from $\boxed{r_m \text{ to } \infty}$ (again, perihelion 2.5 (ii) to final position) is actually same (not opposite as naively thought to (!)) that from $\boxed{\infty \text{ to } r_m}$ (along initial to perihelion position). Thus, we get

$$\theta_{orbit} \text{ for final position} = 2 \times \underbrace{\left[\int_{\infty}^{r_m} \frac{dr'/r'^2}{\sqrt{\frac{2mE}{\ell^2} - \frac{2mV(r')}{\ell^2} - \frac{1}{r'^2}}} \right]}_{\text{same as integral appearing in perihelion condition used before}} + \pi$$

$$= 2 \underbrace{(-\psi)}_{\text{again, see perihelion position discussion}} + \pi$$

$$= \theta \text{ from Eq. (1)}$$

again as expected from figure.

If analytic formula for $\boxed{r(\theta_{orbit})}$ is known, then θ and s can be related easily using it.

For example, Rutherford scattering, i.e., charged particles ($Z'|e|$) repelled by Coulomb field of nucleus ($Z|e|$), so that

force $= ZZ'e^2/r^2$, i.e., inverse square law (like in Kepler problem), but repulsive (vs. attractive for Kepler problem) $\Rightarrow$ results for Kepler problem can be suitably adapted here, with

$\boxed{k = -ZZ'e^2}$, i.e., < 0 (vs. > 0 in Kepler problem) ...(7)

Orbit equation:

$$1/r = \frac{mk}{l^2}\left[1 + \underbrace{\sqrt{1 + \frac{2El^2}{mk^2}}}_{\varepsilon \text{ (eccentricity)}} \cos(\theta_{orbit} - \theta')\right]$$

becomes

$$1/r = \underbrace{\frac{mZZ'e^2}{l^2}}_{>0}\left[-1 - \varepsilon\cos(\theta_{orbit} - \theta')\right] \quad \ldots(8)$$

$(\varepsilon > 1)$

with orbit being hyperbola, since $E > 0$ here.

Using $\boxed{\text{same}}$ convention for θ_{orbit} as in earlier figure, we see that perihelion (smallest r) occurs when [i.e., RHS of Eq. 8 is maximized]

$$\cos(\theta_{orbit} - \theta') = -1 \Rightarrow \underbrace{\pi - \psi}_{\text{see figure}} - \theta' = \cos\pi \text{ or } \psi = -\theta' \ldots(9)$$

Initial asymptotic position corresponds to $r \to \infty$, $\theta_{orbit} = \pi$ so that Eq. (8) gives

$$-1 - \cos(\pi - \theta')\,\varepsilon = 0 \quad \text{or} \quad \cos\theta' = 1/\varepsilon\,(<1) \ldots (10)$$

Eqs. (9) & (10) give

$$\cos\psi = 1/\varepsilon \quad \text{or [using Eq. (1) and some algebra]}$$

$$\boxed{\cot\theta/2 =} \sqrt{\epsilon^2 - 1} = \boxed{\frac{2Es}{ZZ'e^2}} \ldots (11)$$

[using last of Eq. (3) in formula for ϵ]

[Some quick checks]

(i) $\boxed{s \to \infty}$ (fixed E) should give very small deflection, since particle doesn't really feel central force, i.e., $\theta \to 0$ ... in agreement with Eq. (11)

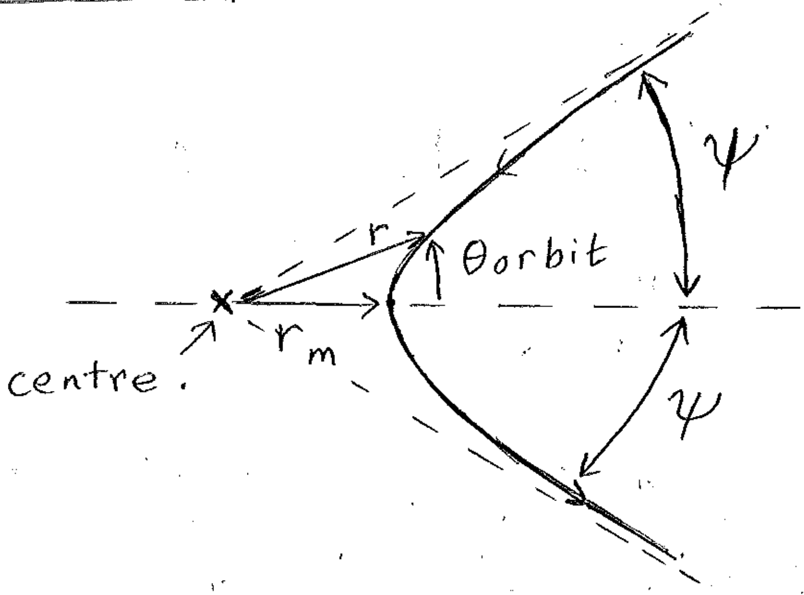
Similarly, $\boxed{s = 0}$, i.e., particle headed straight toward centre of force will be eventually turned around (no matter what its E is), since $V \to \infty$ as $r \to 0$ (particle approaches centre) ... so that $\theta = \pi$ (and using figure), again seen in Eq. (11)

(ii) In Eq. (8), $\boxed{\theta_{orbit} = \theta}$ (i.e., final asymptotic position of particle) should also give $r \to 0$: indeed plugging $\theta = \pi - 2\psi$ [as in Eq. (1)], along with $\theta' = -\psi$ [as in Eq. (9)] in Eq. (8), we find

$$1/r \propto [-1 - \varepsilon\cos(\pi - \psi)] = (-1 + \varepsilon\cos\psi) \to 0 \quad (\text{using } \cos\psi = 1/\varepsilon)$$

Clearly, we have $\boxed{(\pi - 2\psi) \leq \theta_{orbit} \leq \pi}$, which ensures $\boxed{1/r \geq 0}$ from Eq. (8)!

$\boxed{\text{Alternately}}$, as suggested in GPS,



we can "rotate" earlier figure to make it look like above, i.e., perhaps a more standard one, where perihelion is $\boxed{\theta_{orbit} = 0}$ (cf. figure on 1st page). Then, Eq. (8) gives (maximizing $1/r$) $\cos(\theta_{orbit} - \theta') = -1$ [as in 1st of Eq. (9), but now with $\theta_{orbit} = 0$ at perihelion by convention] $\Rightarrow$ we must choose $\boxed{\theta' = \pi}$

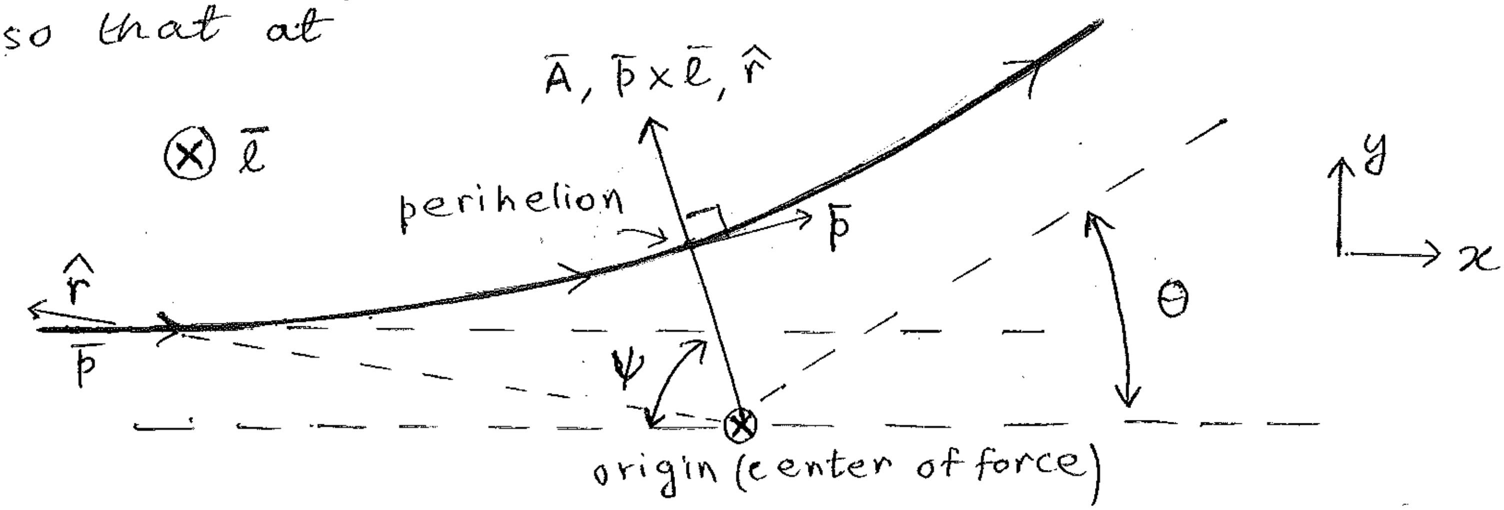
On the other hand initial (or final) position of particle ($r \to \infty$) corresponds to $\boxed{\theta_{orbit} = \pm \psi}$. Plugging this in Eq. (8) [with $\theta' = \pi$ as above] gives

$$\cos\psi = 1/\varepsilon$$ as before...

— A $\boxed{3^{rd}(!)\ \text{derivation}}$ using $\boxed{\text{Laplace-Runge-Lenz vector}}$ ($\bar{A}$) is actually the simplest: recall

$\bar{A}$ (= constant of motion) = $\boxed{\bar{p} \times \bar{l} - mk\hat{r}}$, with $\bar{l} = \bar{r} \times \bar{p}$

We have $\boxed{\bar{\ell}}$ going into plane of figure below so that at



(a) $\boxed{\text{initial}}$ position (asymptotically), we have

$$\bar{A} = (m v_0 \ell)(\hat{y}) - (m k)(-\hat{x}) \quad (\text{with } x, y\text{-axes as shown})$$

initial speed

$$= m v_0 \ell\ \hat{y} - m|k|\ \hat{x} \quad (\text{since } k<0 \text{ here})$$

(b) while at perihelion $\bar{p}\times\bar{\ell}$ is along $\hat{r}$, thus so is $\bar{A}$, i.e, at an angle $(\pi-\psi)$ relative to x-axis

— Equating directions of (a) & (b) [again $\bar{A}$ is constant of motion], we get

$$-\frac{m|k|}{m v_0 \ell} = \frac{\cos(\pi-\psi)}{\sin(\pi-\psi)} = -(\tan\psi)^{-1}$$

Using $E = \frac{1}{2} m v_0^2$ and $\varepsilon^2 = 1 + 2E\ell^2/(mk^2)$, we then obtain

$$\left(1+\tan^2\psi\right) = \frac{1}{\cos^2\psi} = 1 + \underbrace{\left[\frac{k^2}{\ell^2 (2E/m)}\right]}_{= \varepsilon^2}$$

i.e., $\cos\psi = 1/\varepsilon$ as before

— Again, $\boxed{\bar{A} \text{ points in direction of position vector of perihelion}}$ for this hyperbolic trajectory (just like for ellipse / hyperbola discussed in context of Kepler problem)

– However, at $\boxed{\text{perihelion}}$, $\boxed{\theta_{orbit} = (\pi + \theta')}$ here [whether we use convention for θ_{orbit} of Eq. (9), i.e., $\theta' = -\psi$, so that θ_{orbit} at perihelion $= (\pi - \psi)$ as in figure on page (1) $\boxed{\text{Or}}$ we choose figure on page 5, i.e., $\theta' = \pi$, so that θ_{orbit} at perihelion $= 2\pi$ "=" 0 as in that figure], cf. for $\boxed{\text{all}}$ (i.e whether elliptical/closed or/hyperbola) orbits for planets/comets, perihelion was $\boxed{\theta_{orbit} = \theta'}$ (kepler problem)

– The reason for above difference is flip of sign of k: again, for Rutherford/repulsive case,

$$1/r = \underset{\substack{\uparrow \\ -mk/\ell^2 = \frac{m Z Z' e^2}{\ell^2}}}{(\text{positive})} \left[-1 - \varepsilon \cos(\theta_{orbit} - \theta') \right]$$

so that $1/r \geq 0$ requires $\cos(\theta_{orbit} - \theta') < -1/\varepsilon$; in particular $1/r$ is maximized (perihelion) for $\cos(\theta_{orbit} - \theta') = -1$, i.e., $\theta_{orbit} = \pi + \theta'$

– The above subtlety is also seen while comparing orbit equation derived using $\bar{A}$ = constant with that by actually solving EOM, i.e.,

$$1/r = \frac{mk}{\ell^2} \left(1 + \frac{A}{mk} \cos \tilde{\theta} \right)$$ angle between $\bar{r}$ & $\bar{A}$

$$= \underbrace{-\frac{mk}{\ell^2}}_{>0} \left[-1 + \underbrace{\left(-\frac{A}{mk} \right)}_{>0} \cos \tilde{\theta} \right] \quad \boxed{\text{vs.}}$$

$$-\frac{mk}{\ell^2} \left[-1 - \varepsilon \cos(\theta_{orbit} - \theta') \right]$$

$$= -\frac{mk}{\ell^2} \left\{ -1 + \varepsilon \cos\left[\theta_{orbit} - (\pi + \theta') \right] \right\}$$

so that matching gives

$\frac{-A}{mK}$ (>0 because $K<0$ now) $= \varepsilon$ and

$\tilde{\theta} = \theta_{orbit} - (\pi + \theta')$ or $\boxed{\bar{A} \text{ is along } (\pi + \theta')}$,

again, position vector of perihelion