Phys 601 by Dr. Agashe

Rigid Body Motion

[mostly Based on chapter 3 of DT and chapters 4, 5 of GPS]

- So far, we studied motion of isolated/independent point particles
- Onto extended objects/rigid bodies, i.e., collection of N points so that distance between any pair is fixed : $|\bar{r}_i - \bar{r}_j| = \text{constant} \; (\forall \, i, j)$, although direction of $\bar{r}_i - \bar{r}_j$ can change
  - we can transition from discrete/set of particles to continuous system simply by
    $$\sum_i m_i \rightarrow \int d\bar{r} \; \underbrace{\rho(\bar{r})}_{\text{density of rigid body}}$$
- A rigid body has 6 degrees of freedom (d.o.f.), 3 of which describe translation, while 3 are rotational
- To start with, we will focus on rotational motion only
- Outline of formalism part
  - kinematics : schematically, $\bar{v}$ (linear velocity) of point particle $\rightarrow \boxed{\bar{\omega}}$ (angular velocity) of rigid body

this note

(Plan : simple/intuitive picture, followed by formal, more mathematical language)

→ $\boxed{\text{next}}$ note

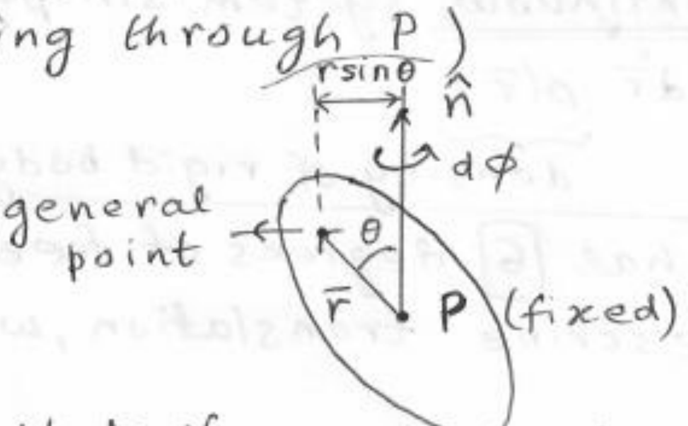
— **dynamics** : for point particle, $\bar{F} = d\bar{p}/dt$, where $\bar{p} = m\bar{v}$ → for rigid body rotation (linear momentum)

$\bar{\tau}$ (torque) $= \dfrac{d\bar{L}}{dt}$, where $L = I\,\bar{\omega}$

angular momentum; moment of inertia (will be generalized to tensor)

— $\boxed{\text{Angular velocity}}$ (section 3.1 of DT) (warm-up)

— Usual / simple language : rigid body fixed at point P (can be taken to be center-of-mass), rotating by (small) angle $d\phi$ about axis $\hat{n}$ (passing through P)



— magnitude of displacement of general point with position vector $\bar{r}$ is given by $|d\bar{r}| = \underbrace{|\bar{r}|\,\sin\theta}_{\text{radius of circle along (small) part of which point moves}}\, d\phi$

and direction of $d\bar{r}$ is $\perp$ to $\bar{r}$ and $\hat{n}$

(again, distance of general point to P is fixed)

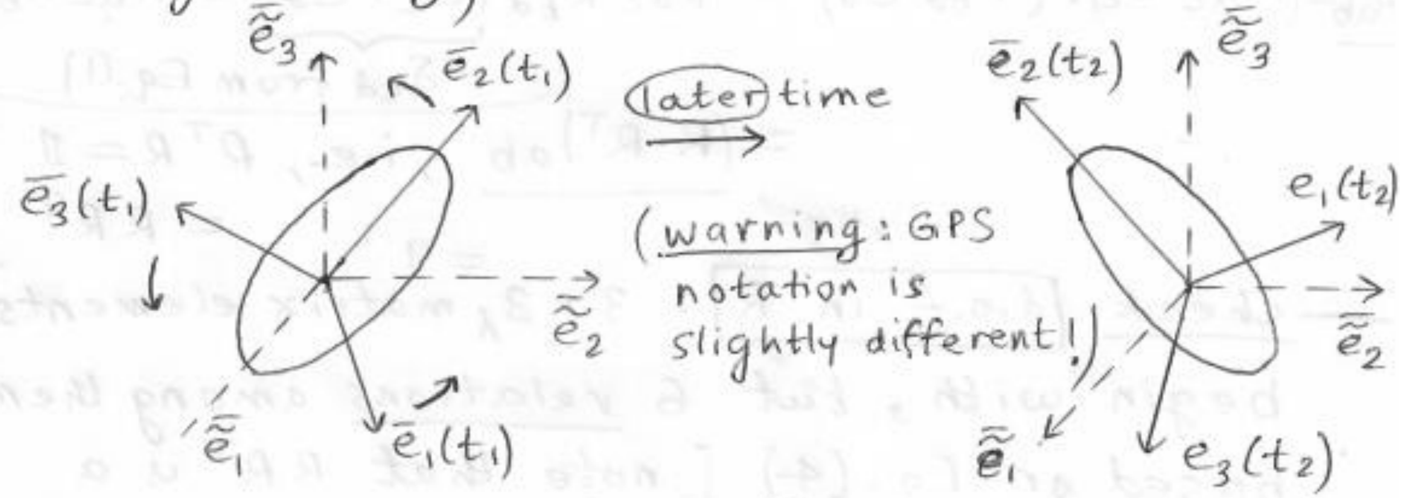
$$\Rightarrow d\bar{r} = \underbrace{d\bar{\phi}}_{\hat{n}\,d\phi} \times \bar{r} \quad \text{or} \quad \dot{\bar{r}} = \underbrace{\bar{\omega}}_{d\bar{\phi}/dt} \times \bar{r} \quad \ldots (1)$$

(usual)

— Tale of 2 frames

— Onto more formal/mathematical notation:

consider $\boxed{\text{two}}$ frames/coordinate systems shown below, i.e., fixed/external to body/space frame denoted by (unit vectors) $\tilde{\bar{e}}_{a=1,2,3}$ and moving/body frame $\bar{e}_a$ (i.e., latter rotates with the rigid body)



[For concreteness, $\bar{e}_a$ or $\tilde{\bar{e}}_a$'s could just be unit vectors along x, y, z axes as shown above.]

— We can always choose both $\boxed{\text{axes}}$ to be orthogonal:

$$\bar{e}_a(t) \cdot \bar{e}_b(t) = \delta_{ab} = \tilde{\bar{e}}_a \cdot \tilde{\bar{e}}_b \quad \ldots (2)$$

moving with body ← time-independent

[Anticipating discussion to come later, there is a natural/convenient choice for body axes such that inertia tensor is diagonal.]

— We can express body frame unit vectors (time-dependent) in terms of space frame via 3×3 matrix, $\boxed{R_{ab}}$:

$$\bar{e}_a(t) = \sum_b R_{ab}(t)\, \underbrace{\tilde{\bar{e}}_b}_{\leftarrow \text{constant}} \quad \ldots (3)$$

— clearly, $\boxed{R}$ is $\boxed{\text{orthogonal}}$, i.e.,

$$R^T R = \mathbb{1} \quad \ldots (4)$$

[Explicitly, we have $\bar{e}_a \cdot \bar{e}_b = \delta_{ab}$ from Eq.(1) so that plugging in Eq.(3) gives

$$\delta_{ab} = (R_{ac}\, \tilde{\bar{e}}_c) \cdot (R_{bd}\, \tilde{\bar{e}}_d) = R_{ac} R_{bd} \underbrace{(\tilde{\bar{e}}_c \cdot \tilde{\bar{e}}_d)}_{\delta_{cd} \text{ from Eq.(1)}} = R_{ac} R_{bc}$$

$$= (R\, R^T)_{ab} \text{, i.e., } R^T R = \mathbb{1} = R R^T \text{]}$$

— check $\boxed{\text{d.o.f. in } R}$: 3×3 $(= 9)$ matrix elements to begin with, but 6 relations among them based on Eq.(4) [note that $R R^T$ is a symmetric matrix so that it has 6 independent elements] $\Rightarrow 9 - 6 = 3$ d.o.f. in R, as expected since R parametrizes rotation of body (frame)

— In other words, we can "replace" moving/body frame by 3×3 orthogonal matrix, $R(t)$ $\Rightarrow$ we $\boxed{\text{expect}}$ angular velocity $(\bar{\omega})$ to be related to $\dot{R}\, (\equiv dR/dt)$

— Let's work out $\boxed{\bar{\omega} - R, \dot{R}}$ relation; what follows is lot of algebra (so, brace yourselves!): end results are in Eqs.(7),(8)

— we can $\boxed{\text{expand / express}}$ position vector of any point in rigid body $\boxed{\text{either}}$ in terms of unit vectors of space/fixed or moving/ body frame:

$$\bar{r}(t) = \begin{cases} \sum_a \tilde{r}_a(t)\, \tilde{\bar{e}}_a & \text{in space/fixed frame} \\ \sum_a r_a\, \bar{e}_a(t) & \text{in moving/body frame} \end{cases} \quad \ldots \boxed{(5)(a)}$$

we'll do analogously for other vectors $\boxed{\text{later}}$

independent of time

Again, coordinates of point relative to body axes are fixed (being a rigid body), but these axes are themselves moving in space frame.

[If you'd like to be concrete, take

$$\bar{r} = \tilde{x}\,\bar{\tilde{i}} + \tilde{y}\,\bar{\tilde{j}} + \tilde{z}\,\bar{\tilde{k}} = x\,\bar{i} + y\,\bar{j} + z\,\bar{k}$$

all are time dependent

where $\bar{i}, \bar{j}, \bar{k}$ are unit vectors along Cartesian axes in body frame (similarly $\bar{\tilde{i}}\ldots$ in spaceframe)]

— note that $\tilde{r}_b(t) = r_a R_{ab}(t)$ as per Eq.(3) and above

— So, Eq.(5) gives

$$\frac{d\bar{r}}{dt} = \begin{cases} \sum_a \left(d\tilde{r}_a/dt\right) \tilde{\bar{e}}_a & \text{in space/inertial frame} \\ \sum_a r_a \underbrace{d\bar{e}_a/dt}_{\sum_b dR_{ab}/dt\, \tilde{\bar{e}}_b \text{ as per Eq.(3)}} & \text{in body frame} \end{cases} \quad \ldots \boxed{(5)(b)}$$

note this is velocity in space frame seen by inertial observer, but can be resolved in either set of axes

— consider how body frame axis itself changes with time (i.e., if you wish, set $r_a = 1$, $r_{b,c} = 0$ above):

$$d\bar{e}_a/dt = \sum_b dR_{ab}/dt\, \tilde{\bar{e}}_b = \sum_{b,c} \frac{dR_{ab}}{dt} \left(R^{-1}\right)_{bc} \boxed{\bar{e}_c}$$

i.e., expressed in bodyframe axes. use inverse of Eq.(3)

$\equiv \sum_c \Omega_{ac} \; \bar{e}_c$ ...(6), where

$\boxed{\Omega_{ac}} \equiv \sum_b \dot{R}_{ab} (R^{-1})_{bc} = \sum_b \dot{R}_{ab} R_{cb}$ ...(7) ← use $R^{-1} = R^T$ from Eq.(4)

— It looks like we have a $\boxed{\text{tensor}}$ thus far, but we can "reduce" it to a vector (i.e., 3 components) by realizing that Ω is antisymmetric:

$$\sum_b R_{ab} \underbrace{R_{cb}}_{(R^T)_{bc}} = \delta_{ac} \xRightarrow{\text{taking derivatives}} \sum_b \underbrace{\dot{R}_{ab} R_{cb}}_{\Omega_{ac}} + \underbrace{R_{ab} \dot{R}_{cb}}_{\Omega_{ca}} = 0$$

use Eq.(4) ... use Eq.(7)

— So, we define a vector, $\boxed{\bar{\omega}_{(formal)}}$ expressed in body frame as

$$\bar{\omega}_{(formal)} = \sum_a \omega_a \bar{e}_a, \text{ where } \omega_a = \sum_{b,c} \tfrac{1}{2} \epsilon_{abc} \Omega_{bc} \; ...(8)$$

[Recall 3×3 antisymmetric matrix has only 3 independent elements.] in order to show (formal) $\bar{\omega}$ is same as usual one

— Then, it is "straightforward" to re-write Eq.(6) as

$$d\bar{e}_a/dt = \sum_{bc} -(\epsilon_{abc} \omega_b) \bar{e}_c = \bar{\omega} \times (\bar{e}_a) \left[\underline{\text{not}} \; (\bar{\omega} \times \bar{e})_a ! \right] \; ...(9)$$

[Check (can skip in lecture): plug $\omega_b = \frac{1}{2} \epsilon_{bde} \Omega_{de}$ from Eq.(8) into middle of Eq.(9) above to get

$$- \epsilon_{abc} \omega_b = -\tfrac{1}{2} \epsilon_{abc} \epsilon_{bde} \Omega_{de} = -\tfrac{1}{2} (\epsilon_{bca} \epsilon_{bde}) \Omega_{de}$$

$$= -\tfrac{1}{2} (\delta_{cd} \delta_{ae} - \delta_{ce} \delta_{ad}) \times \Omega_{de} = -\tfrac{1}{2} (\Omega_{ca} - \Omega_{ac}) = + \Omega_{ac}$$

where standard properties of ϵ have been used.]

— Plug Eq.(9) into 2nd line of Eq.(5)(b) to get

$$\dot{\bar{r}} = \sum_a \underset{\substack{\uparrow \\ \text{constant}}}{r_a}\, \dot{\bar{e}}_a(t) = \sum_a r_a\left[\bar{\omega} \times (\bar{e}_a)\right] = \bar{\omega} \times \underbrace{\sum_a r_a \bar{e}_a}_{\bar{r}}$$ ⑦

i.e., same as Eq. (1) obtained using simple/usual picture. Thus, the 2 definitions of $\bar{\omega}$ [simple used in Eq. (1) and more formal one in terms of Eq. (7), (8)] are indeed identical.

— Onto specific $\boxed{\text{parametrization}}$ of $R_{ab}(t)$:

Euler angles (sec. 3.5 of DT)

— so far, a bit abstract (!); let's give a face to 3 × 3 orthogonal matrix relating 2 frames

— Euler's theorem : arbitrary rotation can be expressed as product of 3 successive rotations (3 angles) about 3 (in general different) axes

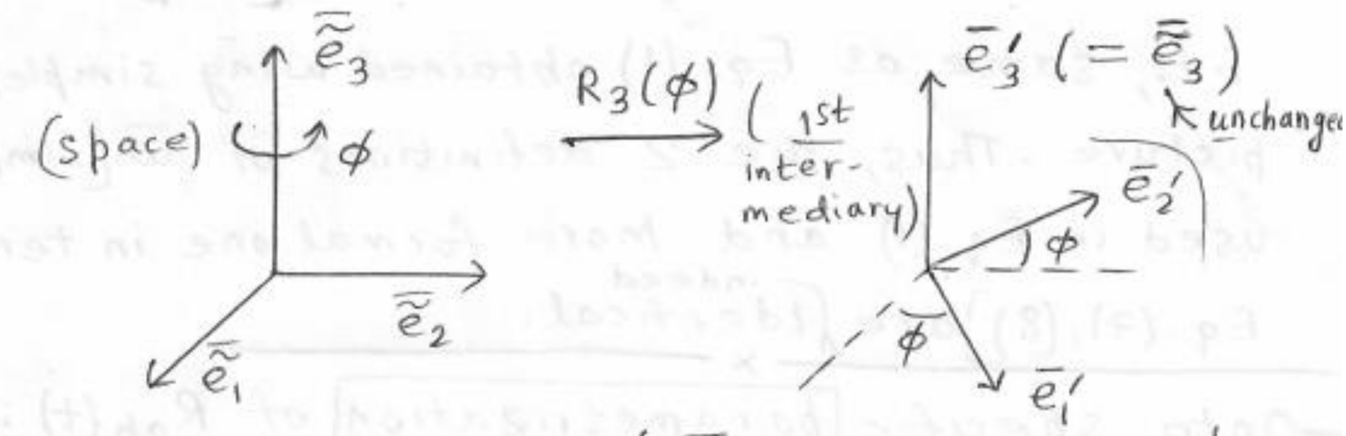
— Again, we want to go from $\tilde{\bar{e}}_a$ (space frame) to $\bar{e}_a$ (body frame) via 3 × 3 orthogononal matrix (R) parametrized by 3 (Euler) angles (as follows, which is lot of $\boxed{\text{algebra}}$; so beware!)

— 3 steps/"component" rotations so that we'll have 2 intermediary (again!) frames: schematically

$$\underset{(\text{space})}{\tilde{\bar{e}}_a} \xrightarrow[\substack{\text{rotate} \\ \text{about 3rd} \\ \text{axis by } \boxed{\phi}}]{R_3(\phi)} \underset{(\text{1st intermediary})}{\bar{e}'_a} \xrightarrow[(\text{1st axis by } \theta)]{R_1(\theta)} \bar{e}''_a \xrightarrow[\substack{\text{3rd axis} \\ (\text{different} \\ \text{than 1st step}) \text{ by } \underline{\Psi}}]{R_3(\Psi)} \underset{(\text{body})}{\bar{e}_a}$$

- In more/gory $\boxed{\text{detail}}$, we have in "z" ⑧

$\boxed{1^{st}}$ step (rotate by ϕ about $\tilde{\bar{e}}_3$ axis):



- clearly (z) component of a vector is unchanged, whereas x, y just get rotated into each other by ϕ so that ...(8)

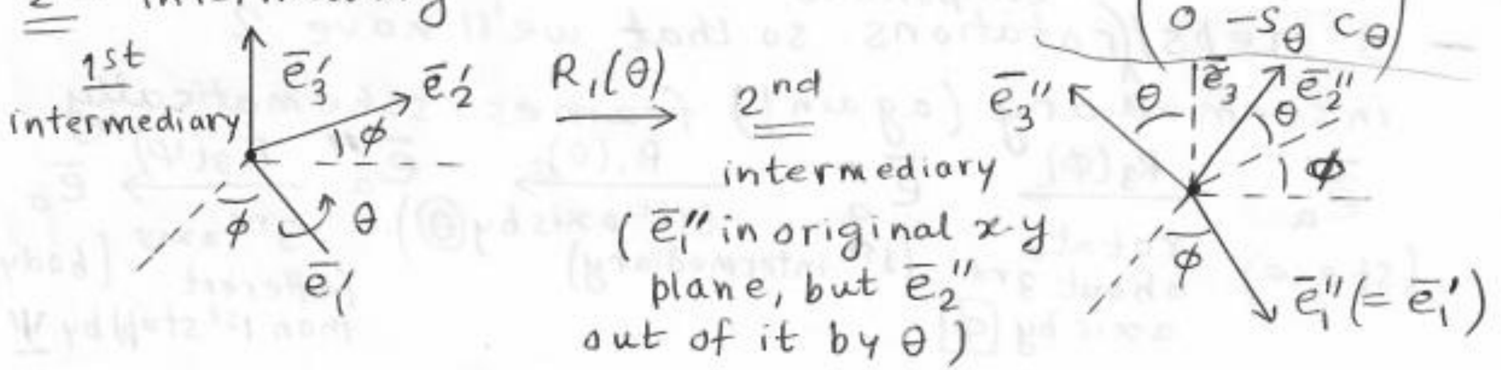
($\bar{e}'_{2,1}$ still in original x-y plane)

$$\underset{\text{1st intermediary}}{\bar{e}'_a} = [R_3(\phi)]_{ab} \underset{\text{space}}{\tilde{\bar{e}}_b}, \text{ where } R_3(\phi) = \begin{pmatrix} c_\phi & s_\phi & 0 \\ -s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(where $c_\phi \equiv \cos\phi$ and $s_\phi \equiv \sin\phi$)

- $\underline{\underline{2^{nd}}}$ step [rotate by θ about (new) x-axis, i.e., $\bar{e}_1'$ (called line of nodes] so that z-component now changes (so does y), but x is unchanged:

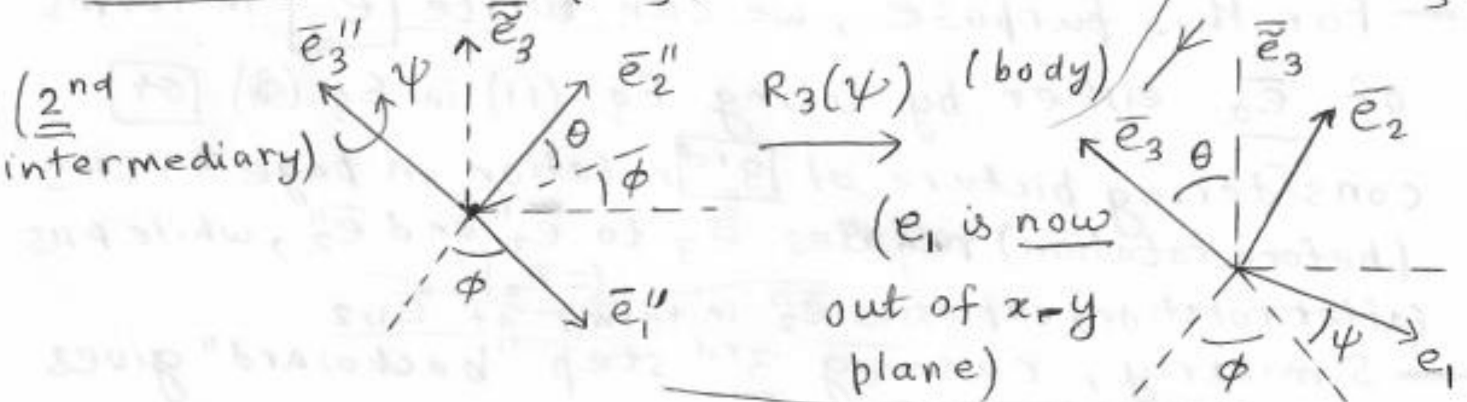
$$\underset{2^{nd}\text{ intermediary}}{\bar{e}''_a} = [R_1(\theta)]_{ab} \underset{1^{st}\text{ intermediary}}{\bar{e}'_b}, \text{ with } R_1(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_\theta & s_\theta \\ 0 & -s_\theta & c_\theta \end{pmatrix} \quad \ldots(9)$$



— $\underline{3^{rd}}$ step [rotation by ψ about (new) z-axis, i.e., $\bar{e}_3''$] so that $\leftarrow$ $\underline{2^{nd}}$ intermediary

$$\underset{(\text{body})}{\bar{e}_a} = \left[R_3(\psi)\right]_{ab}\, \bar{e}''_b, \text{ with } R_3(\psi) = \begin{pmatrix} c_\psi & s_\psi & 0 \\ -s_\psi & c_\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \ldots(10)$$

i.e., "like" $\underline{1^{st}}$ step, but about another z-axis ($\bar{e}_3''$)



— So, putting it all together (in order):

$$R_{ab}(\phi, \theta, \psi) = \left[\underset{3^{rd}\text{ rotation}}{R_3(\psi)}\; R_1(\theta)\; \underset{1^{st}\text{ rotation}}{R_3(\phi)}\right]_{ab} \quad \left[\begin{array}{l}\text{use} \\ \text{Eqs. (8)-(10)}\end{array}\right]$$

$$= \begin{pmatrix} c_\psi c_\phi - c_\theta s_\phi s_\psi & s_\theta c_\psi + c_\theta s_\psi c_\phi & s_\theta s_\psi \\ -c_\phi s_\psi - c_\theta c_\psi s_\phi & -s_\psi s_\phi + c_\theta c_\psi c_\phi & s_\theta c_\psi \\ s_\theta s_\phi & -s_\theta c_\phi & c_\theta \end{pmatrix} \quad \ldots(11)$$

— Finally, let's express $\boxed{\bar{\omega}}$ in terms of Euler angles' time derivatives

— Brute force way: plug Eq. (11) into Eqs. (7), (8)

— Quicker: considering (small) changes $d\phi$, $d\theta$ & $d\psi$ in dt, we get

$$\bar{\omega} = \dot{\phi}\,\bar{\tilde{e}}_3 + \dot{\theta}\,\bar{e}_1' + \dot{\psi}\,\bar{e}_3 \quad \cdots (12)$$

since ϕ was rotation about $\bar{\tilde{e}}_3$ (space)

θ about e_1' (x-axis of intermediaries)

ψ is about e_3(body)

— However, we would like to <u>express</u> $\boxed{\bar{\omega}}$ in <u>terms of (only) $\boxed{\text{body}}$ frame unit vectors</u>

— For this purpose, we can write $\boxed{\bar{\tilde{e}}_3}$ in terms of $\underline{\bar{e}_a}$ either by using Eq.(11) in Eq.(3) $\boxed{\text{or}}$ considering picture of $\boxed{3^{rd}}$ rotation on page 9: LHS (before rotation) relates $\bar{\tilde{e}}_3$ to $\bar{e}_3''$ $(= \bar{e}_3)$ and $\bar{e}_2''$, while RHS (after rotation) expresses $\bar{e}_2''$ in terms of $\bar{e}_{1,2}$

— Similarly, running $\underline{3^{rd}}$ step "backward" gives e_1' $(= \bar{e}_1'')$ in terms of $\bar{e}_{1,2}$ (end of 3^{rd} step)

e_1': end of 1^{st} step

$\bar{e}_1''$: end of 2^{nd} step

— <u>Rest</u> is left for <u>HW $\boxed{6.3}$</u> (a)

— Similarly, we can express $\bar{\omega}$ in terms of $\bar{\tilde{e}}_a$ (i.e., <u>space</u>-frame axes): see <u>HW $\boxed{6.3}$</u>(b)