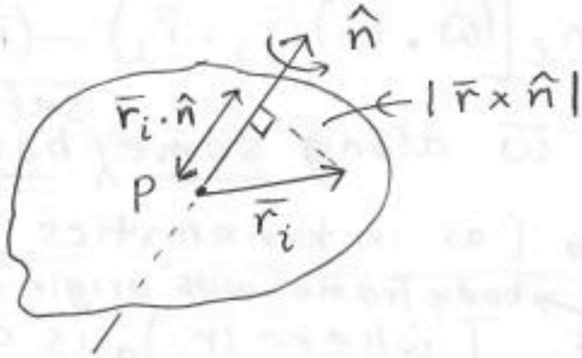
Rigid Body motion (continued)

— After basics of kinematics, onto dynamics, i.e., analog of $\bar{F} = d\bar{p}/dt$, with $p = m\bar{v}$ for particles : $\bar{\tau}$(torque) $= \frac{d\bar{L}}{dt}$, with $L = I\bar{\omega}$, but suitably generalized to handle arbitrary rotations

Plan:
— start with (generalization of) moment of inertia (I), i.e., analog of m; then consider $\bar{L}$ (angular momentum) in analogy with $\bar{p} = m\bar{v}$

Review
— simple/usual idea of moment of inertia :



- consider the usual case of rigid body rotating with angular velocity $\bar{\omega}$ about axis $\hat{n}$ (unit vector), with P being (fixed) point on axis

— define I(usual) about axis $\equiv \sum_{i \leftarrow \text{particles in body}} m_i (\perp \text{ distance from axis})^2$ ... (1)

$$= \sum_i m_i (\bar{r}_i \times \hat{n}) \cdot (\bar{r}_i \times \hat{n}) \text{ (see figure)}$$

$$= \sum_i m_i/\omega^2 \, (\bar{r}_i \times \bar{\omega}) \cdot (\bar{r}_i \times \omega) \quad (\bar{\omega} = \omega\hat{n})$$

$= \sum m_i/_{\omega^2}\, v_i^2$, where $\bar{v}_i = \dot{\bar{r}}_i = \bar{r}_i \times \bar{\omega}$
(from kinematics)

$= \frac{2}{\omega^2} T$, where T is total kinetic energy (KE)

i.e., $T = \frac{1}{2}\omega^2 I(\text{usual}) \ldots (2)$

(as is well known!)

— Onto $\boxed{\text{generalization}}$ to inertia tensor $\rightarrow$ $\boxed{\text{defined}}$ about point P in body
(again, in order to deal with complicated, arbitrary rotations vs. simple studied earlier, e.g., in undergraduate courses) using $\boxed{KE}$ as in last lines above, i.e., $\rightarrow$ again, velocity as constructed/defined by inertial observer

$$T = \tfrac{1}{2}\sum_i m_i |\dot{\bar{r}}_i|^2 = \tfrac{1}{2}\sum_i m_i (\bar{\omega}\times\bar{r}_i)\cdot(\omega\times\bar{r}_i)$$

$$= \tfrac{1}{2}\sum_i m_i\left[(\bar{\omega}\cdot\bar{\omega})(\bar{r}_i\cdot\bar{r}_i) - (\bar{r}_i\cdot\bar{\omega})^2\right]$$

— Now, resolve $\bar{\omega}$ along some set of body axes, i.e.,

$\bar{\omega} = \sum_{a=1,2,3} \omega_a \bar{e}_a$ (as in kinematics note), with $\rightarrow$ body frame with origin at P

$\bar{r}_i = (r_i)_a\, \bar{e}_a$ [where $(r_i)_a$ is constant, but depends on choice of body axes]

position vector relative to point P in body above

— Then, KE can be written as

again, w.r.t. point in body

$$T = \tfrac{1}{2}\,\omega_a I_{ab}\,\omega_b \equiv \frac{1}{2}\,\bar{\omega}\cdot\boxed{\overleftrightarrow{I}}\cdot\bar{\omega} \;\ldots (3) \quad \hookrightarrow \text{tensor}$$

where $\boxed{I_{ab}} \equiv \sum_i m_i\left[\overbrace{(\bar{r}_i\cdot\bar{r}_i)}^{\sum_c (r_i)_c (r_i)_c}\delta_{ab} - (r_i)_a (r_i)_b\right] \ldots (4)$

— clearly, $\overleftrightarrow{I}$ is time-independent and symmetric

[Obviously if we resolve $\bar{\omega}, \bar{r}$ along space axes instead, i.e, $\bar{\omega} = \tilde{\omega}_a \bar{\tilde{e}}_a$ and $\bar{r}_i = (\tilde{r}_i)_a \bar{\tilde{e}}_a$, then we would have a similar formula for T, but with $\omega_a \to \tilde{\omega}_a$ and $I_{ab} \to \tilde{I}_{ab} = \sum_i m_i \left[\delta_{ab} \sum_c (\tilde{r}_i)_c^2 - (\tilde{r}_i)_a (\tilde{r}_i)_b\right]$

— However, $(\tilde{r}_i)_a$'s are changing with time so that $\tilde{I}_{ab}$ is not constant, hence not so useful]

— Let's make contact with $\boxed{\text{usual}}$ (I) : if $\bar{\omega} = \omega \hat{n}$ (as usual), then Eq.(3) gives

$$T = \omega^2/_2 \; \hat{n} \cdot \overleftrightarrow{I} \cdot \hat{n} = \frac{1}{2}\omega^2 I_{axis} \text{(from tensor)}, \quad \ldots (5)$$

where $I_{axis} = \sum_i m_i \left[|\bar{r}_i|^2 - (\bar{r}_i \cdot \hat{n})^2\right]$ from Eq.(4)

— Comparing Eqs. (2) & (5), we would "like to" $\boxed{\text{identify}}$ → see Eq.(1)

$$I \text{ (usual)} = I_{axis} \text{ (from tensor)} \ldots (6)$$

— Well, let's check Eq.(6) explicitly: we see (from figure) that $\sqrt{\left[|\bar{r}_i|^2 - (\bar{r}_i \cdot \hat{n})^2\right]}$ is $\perp$ distance from point i to axis of rotation so that Eq.(6) is indeed correct

— In other words, $\overleftrightarrow{I}$ (tensor) → defined about P "sandwhiched" between axis of rotation (i.e, dotted into those unit vectors) gives usual moment of inertia (I) about that axis of rotation or usual I is but one "component" of $\overleftrightarrow{I}$

— Now, $\overleftrightarrow{I}$ depends not only on point P in body, but also choice of body axes (P being origin): again $(r_i)_a$'s in $\overleftrightarrow{I}$ formula are body-frame coordinates

— So, <u>in general</u>, $\overleftrightarrow{I}$ is <u>off-diagonal</u>: it can be diagonalized by an (constant) orthogonal transformation (given that $\overleftrightarrow{I}$ is symmetric): schematically

$$O^T \overleftrightarrow{I} O = \overleftrightarrow{I}' \text{ (diagonal), where } O^T O = \mathbb{1}$$

— How to "implement" above O: well, it's (again) 3×3 orthogonal matrix, like R seen in kinematics note for going from space to body frame

— So, here we can $\boxed{\text{rotate}}$ <u>body axes</u> by above O:

$$\underset{\text{new body axes}}{\bar{e}_{\text{diag}}} = O \underset{\text{original body axes}}{\bar{e}}$$

$\Rightarrow$ in new body frame, inertia tensor <u>is</u> diagonal

— Such body axes are called $\boxed{\text{principal}}$ axes: $\overleftrightarrow{I}$ is still in general <u>non-universal</u>, i.e., $\not\propto \mathbb{1}$, say

$$\overleftrightarrow{I} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$

, where I_i's are moments of inertia about those 3 (principal) axes

(Again, $\overleftrightarrow{I}$, i.e., tensor, neatly combines many usual I's about axis)

- For continuous bodies, we get

$$\overleftrightarrow{I} = \int \underbrace{\rho(\bar{r})\, d^3 r}_{dm} \begin{pmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{pmatrix} \quad \ldots (7)$$

Eq.(4) gives $(a = 3, b = 1)$ $(0 - zx)$

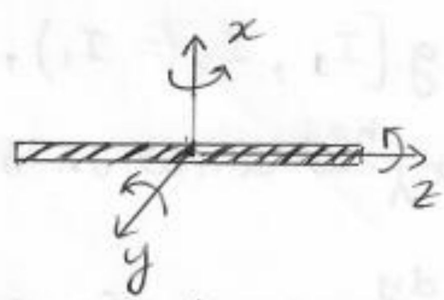
Eq.(4) gives $(a = b = 3 \text{ or } z)$ $(x^2+y^2+z^2-z^2)$

- $\boxed{\text{Examples}}$ of $\overleftrightarrow{I}$ (maybe 1 is enough)

(these results are known from undergraduate, but here derive formally using tensor)

(1) Rod (thin) of mass M, length $\ell \Rightarrow \rho$ (actually linear mass density $= \frac{M}{\ell}$)

$\overleftrightarrow{I}$ about centre and body axes shown:



- By symmetry, the above are $\boxed{\text{principal}}$ axes, i.e.,

$$\overleftrightarrow{I} = \text{diag}\left[\underset{I_{xx}}{I_1}, \underset{I_{yy}}{I_2 (= I_1)}, 0\right]$$

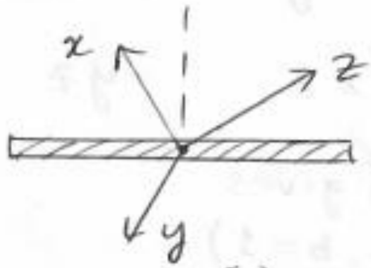
rod along z-axis $\Rightarrow x = y = 0$ for all points $\Rightarrow I_{zz}$ in Eq.(7) $= 0$ due to $x = y = 0$

[can check easily explicitly that off diagonal $\overleftrightarrow{I} = 0$]

$$I_1 = \int_{-\ell/2}^{+\ell/2} \underbrace{\left(\frac{M}{\ell}\right)}_{\rho} z^2 dz = \frac{1}{12} M \ell^2$$

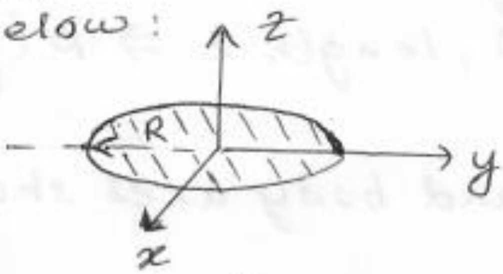
(again, $\overleftrightarrow{I}$ "combines" various I about axes)

[Just to belabor the point about choice of axes, we could have chosen different axes, even with $\boxed{\text{center}}$ of rod as origin, e.g.



which would make $\overleftrightarrow{I}$ off-diagonal, i.e., symmetry makes it clear that the ones before are principal axes.]

(2) Disc of radius R and mass M, with axes/origin as below:



$z = 0$ for all points on disc

Again, $\overleftrightarrow{I} = \text{diag}\left[I_1, I_2 (= I_1), I_3\right]$ by symmetry

[e.g. $I_{xy} \propto \int xy\, dy\, dx$ has cancellation between (x,y) & $(x,-y)$ etc.]

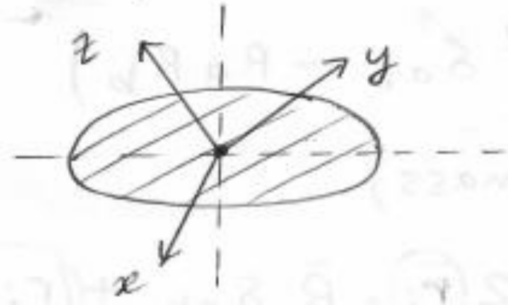
$I_1 = \int \rho \boxed{y^2}\, dx\, dy = I_2 = \int \rho\, dy\, dx\, x^2$, while

$I_3 = \int \rho (x^2+y^2)\, dx\, dy = I_1 + I_2 = 2 I_1$

$= \underbrace{2\pi}_{\int d\theta} \rho \int_0^R r^3\, dr = \frac{1}{2} M R^2 \Rightarrow I_1 = \frac{1}{4} M R^2$

switch to polar coordinates

[Again, $\overleftrightarrow{I}$ <u>not</u> diagonal for other choices of axes through center, e.g.,



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$\boxed{\text{Parallel axis theorem}}$

— <u>Goal</u>: suppose Ⓟ is $\boxed{\text{COM}}$ and we know $\overleftrightarrow{I}$ about it already what if we want instead $\overleftrightarrow{I}$ about $\boxed{\text{another}}$ point Ⓟ′? Should we (re-)calculate using Eq. (7)?

<u>No</u> $\boxed{\text{need}}$, since simple to use theorem:
[again, this is known in context of moment of inertia (about axis), but here work in terms of tensor]

if P′ displaced from $\bar{P}$ (COM) by $\bar{R}$, then

$$(I_R)_{ab} = (I_{COM})_{ab} + M\underbrace{\left(R^2 \delta_{ab} - R_a R_b\right)}\ \ldots (8)$$

about P′, but axes $\boxed{\text{parallel}}$ to original

known (about COM)

tensor form: $\begin{bmatrix} (R^2 - R_x^2) & (-R_x R_y) & \cdots \\ \cdots & R^2 - R_y^2 & \cdots \\ \cdots & \cdots & R^2 - R_z^2 \end{bmatrix}$

[again, can "skip" integral in Eq. (7)]

i.e., no "rotation" to "mix-up" a & b

$\boxed{\text{Proof}}$: $(I_R)_{ab}$ from Eq. (4)

$$= \sum_i m_i \left[|\underbrace{\bar{r}_i - \bar{R}}_{\text{new } \bar{r}_i}|^2 \delta_{ab} - (r_i - R)_a (r_i - R)_b \right]$$

axes <u>parallel</u> <u>crucial</u> here

$$= \underbrace{\sum_i m_i \left[|\bar{r}_i|^2 \delta_{ab} - (r_i)_a (r_i)_b \right]}_{(I_{COM})_{ab}}$$

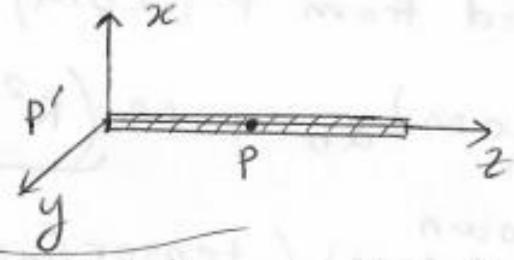
$$+ \underbrace{\sum_i m_i}_{M \text{ (total mass)}} \left(R^2 \delta_{ab} - R_a R_b \right)$$

$$+ \sum_i m_i \left[-2 \bar{r}_i \cdot \bar{R}\, \delta_{ab} + (r_i)_a R_b + (r_i)_b R_a \right]$$

linear in r_i $\Rightarrow$ vanish (only) if $\bar{r}_i$ measured from COM, since $\sum_i m_i \bar{r}_i = 0$ by definition of COM

giving Eq.(8)

— Back to $\boxed{\text{rod}}$: $\overleftrightarrow{I}$ about its end from $\overleftrightarrow{I}$ about center computed earlier:



i.e., $\bar{R} = (0, 0, \ell/2)$ so that

$$I_1(\text{new}) \left[= I_2(\text{new}) \right] = I_1(\text{old}) + M(\ell/2)^2$$

still have that symmetry

$$= \tfrac{1}{3} M \ell^2$$

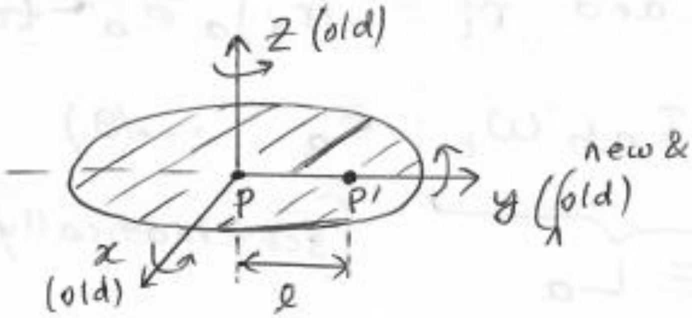
2nd term in Eq.(8) is

$$M^2 R^2 \left[\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} - \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix} \right]$$

[check: should be $\int_0^\ell \rho z^2 dz = M/\ell \; \ell^3/3$], whereas $I_3(\text{new}) = \underset{(=0)}{I_3(\text{old})} + 0 = 0$ (as expected)

$\boxed{\text{Or}}$ back to disc, but about P' with shift from center,

$\bar{R} = (0, \ell, 0)$



— We $\boxed{\text{expect}}$ $I_2 (= I_{yy})$ to be unchanged, while $I_1 (= I_{xx})$ & $I_3 (= I_{zz})$ will be modified

— check: from Eq. 8

$$I_R = M \begin{pmatrix} \frac{1}{4}R^2 & & \\ & \frac{1}{4}R^2 & \\ & & \frac{1}{2}R^2 \end{pmatrix} \leftarrow \text{about COM} \; + \; M\ell^2 \left[\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} - \begin{pmatrix} 0 & & \\ & 1 & \\ & & 0 \end{pmatrix} \right]$$

$$= M \begin{pmatrix} \frac{1}{4}R^2 + \ell^2 & & \\ & \frac{1}{4}R^2 & \\ & & \frac{1}{2}R^2 + \ell^2 \end{pmatrix}$$

×

— Now that we have studied $\boxed{\text{analog of}}$ m, let's move onto that of $\bar{p}$, i.e., $\boxed{\text{angular momentum}}$ about point P in body which is fixed (as before). We have

$$\bar{L} = \sum_i m_i \left(\bar{r}_i \times \dot{\bar{r}}_i \right)$$

again, velocity (time-derivative) as seen by $\boxed{\text{space}}$ frame observer (observer fixed in body will find particle i not moving at all!)

$$= \sum_i m_i \bar{r}_i \times (\bar{\omega} \times \bar{r}_i)$$

$$= \sum_i m_i \left[|\bar{r}_i|^2 \bar{\omega} - (\bar{\omega} \cdot \bar{r}_i) \bar{r}_i \right]$$

We can resolve/express $\bar{\omega}$, $\bar{r}$ in terms of body-frame unit vectors, i.e., as usual $\bar{\omega} = \omega_a \bar{e}_a$ and $\bar{r}_i = (r_i)_a \bar{e}_a$ to give (10)

$$\bar{L} = \sum_a \underbrace{\sum_b I_{ab}\,\omega_b}_{\equiv L_a} \bar{e}_a \quad \ldots (9)$$

$\left[\text{schematically} : \bar{L} = \overleftrightarrow{I} \cdot \bar{\omega}\right]$

where I_{ab} is inertia tensor of Eq. (4), i.e.,

$$I_{ab} = \sum_i m_i \left[|\bar{r}_i|^2 \delta_{ab} - (r_i)_a (r_i)_b \right]$$

[So, angular momentum can actually be "used" <u>instead</u> of kinetic energy to <u>first</u> <u>define</u> I_{ab}]

[Again, once $\bar{L}$ (or velocity of particle) is constructed as above, i.e., by space-frame observer, it can also be resolved along space axes, i.e., $\bar{L} = \sum_a \tilde{L}_a \tilde{\bar{e}}_a$, where $\tilde{L}_a = \sum \tilde{I}_{ab} \tilde{\omega}_b$, but $\tilde{I}_{ab} = \sum_i m_i \left[\delta_{ab} \sum_c (\tilde{r}_i)_c^2 - (\tilde{r}_i)_a (\tilde{r}_i)_b \right]$ is time-dependent, as we discussed in context of kinetic energy also.]

— Note that only for simple cases, i.e., if <u>axis of rotation</u> (i.e., $\bar{\omega}$) is along one of <u>principal</u> axes, then $\boxed{\bar{L} \propto \bar{\omega}}$ (since $\overleftrightarrow{I}$, i.e., tensor, is diagonal)

— In $\boxed{\text{general}}$, $\bar{L}$ is then <u>not</u> "proportional to" $\bar{\omega}$: again, $\bar{\omega}$ need not be along one of principal axis so that $\overleftrightarrow{I}$ defined using $\bar{\omega}$ (and 2 axes $\perp$ to it) is <u>off</u>-diagonal