Phys 601 Dr. Agashe

Heavy, symmetric top ($I_1 = I_2 \neq I_3$): Specific cases (based on GPS section 5.7 and DT section 3.6)

(1). Uniform precession ($\dot{\phi}$ = constant), without nutation ($\dot{\theta} = 0$), like free top

— We would like to see whether $\theta = \theta_0$ = constant ($\dot{\theta} = 0$) is allowed (inspite of gravity tending to make top fall)

— First of all, plugging above into expression for either p_ϕ or E (being constants) gives $\dot{\phi}$ = constant

— Now, $\dot{u} = -S_\theta \dot{\theta}$ and $\dot{u}^2 = f(u)$ [see previous note] implies that $u_0 \equiv C_{\theta_0}$ is zero of $f(u)$, i.e.,

$$f(u_0) = (1 - u_0^2)(\alpha - \beta u_0) - (b - a u_0)^2 = 0 \quad \ldots (12)$$

— Moreover, we would not want any other value of u to be "allowed", i.e., $f(u) < 0$ for $u \neq u_0$
($= \dot{u}^2$)

(In this way, $u = u_0$ for all time.)

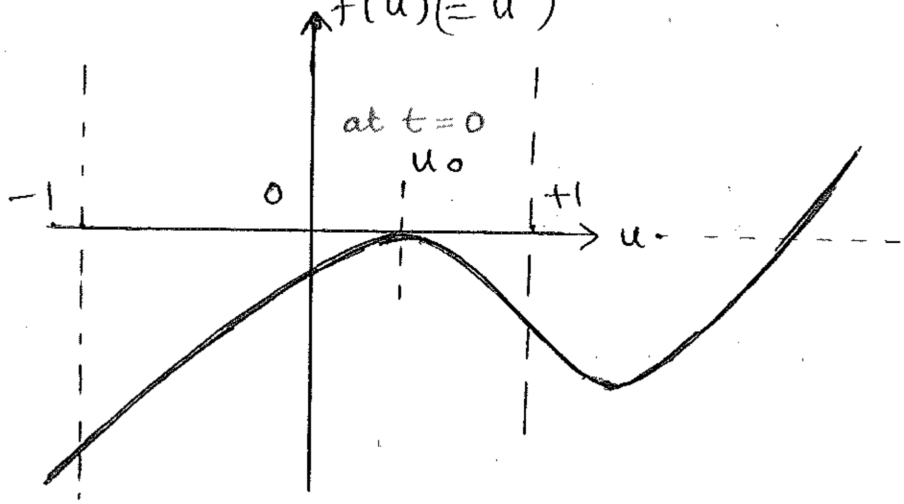
— From figure on page 6 of previous note, it is then clear that u_0 should be a double root of $f(u) = 0$ ($u_1 = u_2 = u_0$): equivalently $f'(u_0) = 0$

[check: for cubic (or higher) polynomial $f(u)$, $f(u_0) = 0$ and $f'(u_0) = 0$ gives

$$f(u) = \underbrace{f(u_0)}_{=0} + \underbrace{f'(u_0)}_{=0}(u - u_0) + f''(u_0)(u - u_0)^2/2! + f'''(u_0)(u - u_0)^3/3!$$

$$= (u-u_0)^2 \left\{ f''(u_0)/2 + f'''(u_0)(u-u_0)/3! \right\}$$

i.e, u_0 is double root of $f(u)=0$]



uniform precession, without nutation

thus, we also have

$$f'(u_0) = 0 = -2u_0(\alpha - \beta u_0) - \beta(1-u_0^2) + 2a(b - au_0) \quad \cdots (13)$$

[This is sort of analogous to $\boxed{\text{circular}}$ orbit for central force corresponding to r_0 being extremum of $V'(r)$.]

similar to relating E, l & r for circular orbit

— Our goal is then the following : given $\boxed{\theta_0}$ and $\boxed{\omega_3}$ (constant spin of top), what's $\boxed{\dot{\phi}}$ needed for this motion? Eqs. (12) & (13) give

[using $\dot{\phi} = (b-au)/(1-u^2)$]

$$\tfrac{1}{2}\beta = a\dot{\phi} - \dot{\phi}^2 u_0$$, i.e., using $I_1 a = I_3 \omega_3$ and $\beta = 2Mgl/I_1$, we have 2 possible values of $\boxed{\dot{\phi}}$ given by $\boxed{\text{quadratic}}$ equation

$$Mgl = \dot{\phi}\left(I_3\omega_3 - I_1\dot{\phi}\cos\theta_0\right) \quad \cdots (14)$$

[Check: $g \to 0$, i.e, $\boxed{\text{free}}$ top in Eq. (14) gives

$$\dot{\phi} = I_3\omega_3/(I_1\cos\theta_0) = \frac{L_3/\cos\theta_0}{I_1} = L/I_1 \text{ as before!}]$$

— Also, <u>existence</u> of solutions for $\dot{\phi}$ requires discriminant > 0, i.e.,

$$(I_3 \omega_3)^2 - 4(-I_1 c_{\theta_0})(-Mgl) > 0 \text{ or}$$

$$\boxed{\omega_3 > 2/I_3 \sqrt{Mgl I_1 c_{\theta_0}}} \quad \ldots (15)$$

i.e., for given θ_0, "top" has to be spinning $\boxed{\text{fast}}$ <u>enough</u> so as to allow uniform precession: for smaller ω_3, top will fall over (i.e., can't support uniform precession without nutation).

— $\boxed{\text{Sanity}}$ check: $\bar{\omega} = \underbrace{\dot{\psi}}_{\text{constant } [=(\omega_3 - \dot{\phi} c_\theta)]} \underbrace{\bar{e}_3}_{\text{moving in space frame since } \phi \neq \text{constant}} + \underbrace{\cancel{\dot{\theta}}}_{0} \bar{e}_1' + \underbrace{\dot{\phi}}_{\text{constant}} \underbrace{\tilde{e}_3}_{\text{fixed (space) axis}}$ $\Rightarrow \bar{\omega} \neq \text{constant}$ in space-frame (as expected from torque $\neq 0$ $\Rightarrow \bar{L} \neq \text{constant}$)

$\boxed{(2)}$. $\boxed{\text{"Sleeping" top}}$

— Suppose we start with a top spinning in upright position, i.e., $\theta = 0$, $\dot{\theta} = 0$ @ $t = 0$ (so that $u = 0$, $\dot{u} = 0$), with $\omega_3 \neq 0$ (at least "blindly")

— Note that we can't really use $\dot{u}^2 = f(u)$ at $t = 0$ since that was derived assuming $(1 - u^2) = s_\theta^2 \neq 0$

— However, at $t = 0$, we have $\bar{e}_3$ (body) $= \tilde{e}_3$ (space) so that $\bar{L} \cdot \bar{e}_3 (= p_\psi) = \bar{L} \cdot \tilde{e}_3 (= p_\phi)$ (see previous note)

i.e., $b (= p_\phi / I_1) = a \left(= \underbrace{I_3 \omega_3}_{p_\psi} / I_1\right)$

[p_ψ, ϕ of course stay the same constants of motion]

— Also, E @ $t = 0$ (but stays same) $= \frac{1}{2} I_3 \omega_3^2 + Mgl$
$(\theta = 0;\ \dot{\theta} = 0\ \&\ \dot{\phi} = 0)$

- Having said in the other note that Euler's equations might not be useful for general discussion of heavy, symmetric top, we will show below that for the specific case of uniform precession without nutation, Euler's equations do give us the condition on $\dot{\phi}$ in Eq. 14
- We get along x (or "1") direction (similary, we can do y or 2):

$$I_1 \dot{\omega}_1 + \omega_2 \omega_3 (I_3 - I_2) = N_1$$, where $\boxed{N_1}$ is component of torque along body x-axis

- clearly $|\bar{N}|$ (magnitude of torque) $= Mgl \sin\theta$ and its direction being along line of nodes (as argued on page 3 of other node), we get

$$N_1 = |N| \sin\theta \cos\psi = Mgl \sin\theta \cos\psi$$

- Also, $\omega_1 = \dot{\phi} s_\theta s_\psi + \dot{\theta} c_\psi$ and $\omega_2 = \dot{\phi} s_\theta c_\psi - \dot{\theta} s_\psi$ give for no nutation (i.e., $\dot{\theta} = 0$) and uniform precession ($\dot{\phi}$ = constant, i.e., $\ddot{\phi} = 0$):

$$\omega_1 = \dot{\phi} s_\theta s_\psi \; ; \; \omega_2 = \dot{\phi} s_\theta c_\psi$$ and

$$\dot{\omega}_1 = \dot{\phi} s_\theta c_\psi \dot{\psi}$$

- Plugging above ω_1, ω_2 and $\dot{\omega}_1$ into Euler's equation gives

$$I_1 \dot{\phi} s_\theta \cancel{c_\psi} \dot{\psi} + \dot{\phi} s_\theta \cancel{c_\psi} \omega_3 (I_3 - \underset{\text{set to } I_1}{I_2}) = Mgl s_\theta \cancel{c_\psi}$$

- Using $\omega_3 = \dot{\psi} + \dot{\phi} c_\theta$ or $\dot{\psi} = \omega_3 - \dot{\phi} c_\theta$ then reduces it to

$$I_1 \dot{\phi} \cancel{S_\theta} (\omega_3 - \dot{\phi} C_\theta) + \dot{\phi} \cancel{S_\theta} \omega_3 (I_3 - I_1) = Mg\ell \cancel{S_\theta}$$

i.e., $\boxed{Mg\ell = \dot{\phi}(I_3 \omega_3 - I_1 \dot{\phi} C_\theta)}$, which is

same as Eq. 14 [with $\theta = \theta_0$ (constant)] so that $E' = E - \frac{1}{2} I_3 \omega_3^2 = Mgl$, i.e.,

$$\alpha \left(= 2E'/I_1\right) = \beta \left(= 2Mgl/I_1\right)$$

— Using $a = b$ & $\alpha = \beta$ from above, we get (now for $\theta \neq 0$!) $f(u) \rightarrow (1-u^2)\,\alpha(1-u) - a^2(1-u)^2$

$$= (1-u)^2 \left[\alpha(1+u) - a^2\right]$$

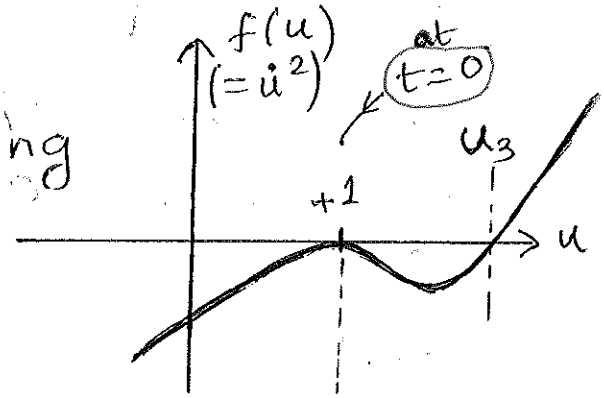
i.e., double root at $u = 1$ [so that $f'(1) = 0$]

and 3rd root is at $u_3 = \left(a^2/\alpha - 1\right)$

— Thus, we have $\boxed{2}$ possibilities:

(a) $\boxed{u_3 > 1}$ so that only physically allowed value is $u = 1$ [i.e., it's like case (1) above with $u_0 \rightarrow 1$] :

(stable) spinning upright



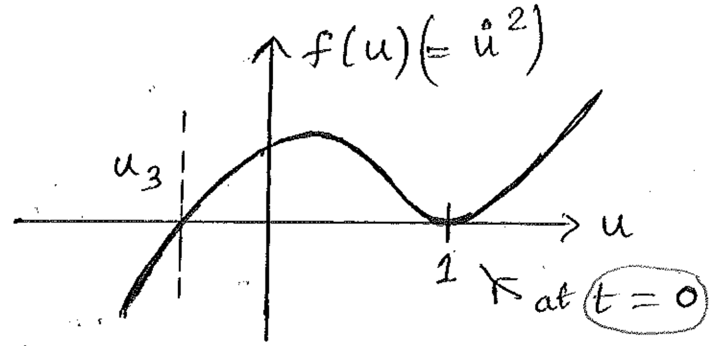
top is "sleeping"

— Thus, top continues to spin upright (again $\boxed{\text{stable}}$ position) : this $\boxed{\text{requires}}$ $a^2/\alpha - 1 > 1$ (u_3 above > 1)

so that $\boxed{\omega_3^2 > 4 I_1 Mgl / I_3^2}$ corresponding to uniform precession case with $\boxed{\theta_0 = 0}$ (as expected).

(b) $\boxed{u_3 < 1}$ (or $\omega_3^2 < 4 I_1 Mgl/I_3^2$) so that other values of $u (< 1)$ are allowed [again, $u = 1$ is still double root or local minimum of $f(u)$]

Unstable spinning upright



— Thus, top is "allowed to" nutate between $\theta = 0$ ($u = 1$) and $\theta = \theta_3$ (corresponding to u_3)

— This looks like special case of "letting top go" (see $\boxed{\text{below}}$): as θ increases ($u = \cos\theta$ reduces), $\dot{\phi}$ must "turn-on" etc.

— However, the difference is that in present case $\boxed{\theta = 0}$ can remain so [even if it is unstable], cf. below [$\theta(@\, t=0) \neq 0$] where θ has to increase: that argument (see below) is based on non-zero torque, but for $\theta = 0$ ($@\, t = 0$), there is $\boxed{\text{no}}$ torque (since gravity — always vertical! — is now through top-axis itself.), which is $\boxed{\text{why}}$ $\theta = 0$, $\dot{\theta} = 0$ remain that way [another argument is $\boxed{\text{below}}$]

since other θ's allowed

— In practice, even if we start with $\omega_3^2 > 4 I_1 Mgl / I_3^2$ (i.e., $\theta = 0$ is stable), what will happen is that friction will reduce ω_3 below above critical value. Subsequently, any small disturbance (e.g., air resistance) will create instability, i.e., sleeping top "wakes up"!

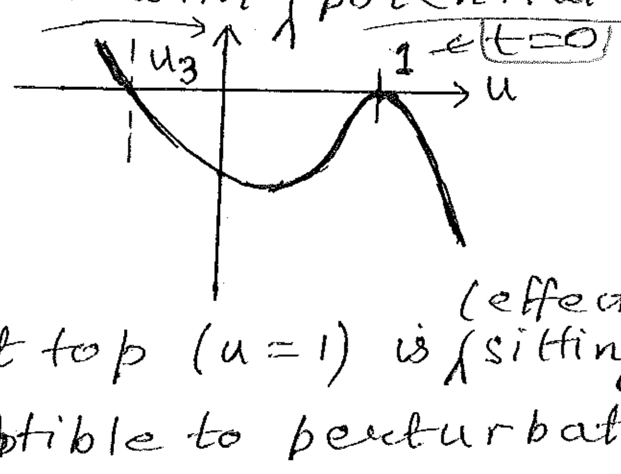
In equations form, we have $\dot{u} = \sqrt{f(u)}$ so that $\ddot{u} = \frac{f'(u)}{2\sqrt{f}}\dot{u} = \frac{f'(u)}{2}$ [for $\dot{u} \neq 0$, just take limit suitably]. So, $\ddot{u}(t=0) = f'(1)/2 = 0$; similarly, higher time derivatives vanish $\Rightarrow \boxed{\theta = 0 \text{ for } t > 0}$ also.

— In fact, above case line of argument $\boxed{\text{explicitly}}$ shows why this is unstable (cf. just saying earlier that other θ's or u's are "allowed" so that top will make an excursion there if perturbed).

— Namely, suppose some perturbation causes θ to deviate from 0 at an infinitesimal time after $t=0$, i.e., $\underbrace{u=1}_{\text{at } t=0} \Rightarrow \underbrace{u=(1-\delta)}_{\text{at } t=0+\Delta t}$ (where $\delta>0$ & $|\delta| \ll 1$). Then above relation, i.e., $\ddot{u} = f'(u)/2$ and looking at plot of $f(u)$, we get $\ddot{u}\,[\text{at } u=(1-\delta)] < 0$, i.e., $\dot{u}$ tends to decrease. But $\dot{u}(t=0 \text{ or } u=1)$ was 0. So, $\dot{u}(\text{at } u=1-\delta) = -\epsilon$ ($\epsilon>0$, $|\epsilon| \ll 1$), i.e., $\boxed{u}$ tends to even more decrease at $t>0$ (or θ increases) so that top continues to fall, i.e, it is an unstable position.

— Equivalently, $\ddot{u} = f'(u)/2 = -\frac{d}{du}\left[\frac{1}{2}f(u)\right]$ suggests that the motion of top is "like" that of a 1 D particle with (effective) potential energy $\left[-\frac{1}{2}f(u)\right]$.

$\left[-\frac{f}{2}(u)\right]$



(effective potential for u)

unstable upright top

— So, upright top ($u=1$) is (effectively) sitting on "top of a hill", i.e., highly susceptible to perturbation (even if it is in equilibrium, i.e., can stay put if not disturbed).

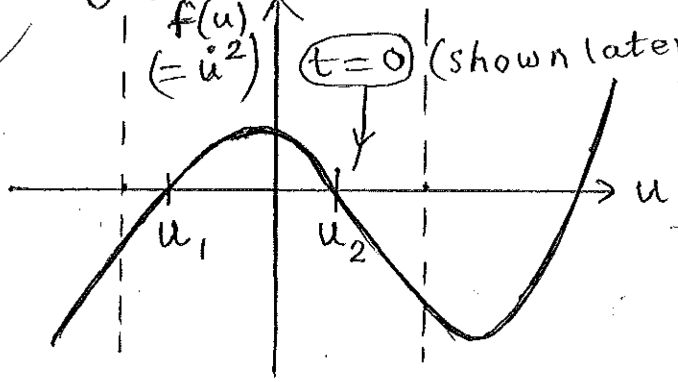
[Whereas, stable top of figure on page 4 is at bottom of hill] [Similarly, for figure on page 6 next.]

(3). $\boxed{\text{Letting the top go}}$ suppose we spin the top ($\omega_3 \neq 0$) and "let it go" at some angle $\theta (\neq 0)$ at $\boxed{t=0}$, with <u>no</u> initial nutation or precession, i.e., $\boxed{\dot{\theta} = \dot{\phi} = 0 \text{ at } t=0}$.
We <u>expect</u> top to <u>fall</u> due to gravity, i.e., θ will reduce: let's see using equations whether this happens.

— First of all, $\dot{u}(t=0) = -\underbrace{S_\theta(t=0)}_{\neq 0}\underbrace{\dot{\theta}(t=0)}_{0} = 0$

$\Rightarrow f[u(t=0)] = 0$, i.e., $u(t=0)$ is <u>either</u> $\boxed{u_1 \text{ or } u_2}$, one of $\boxed{2}$ turning <u>points</u> (see figure below).
(<u>zeroes of f</u>)

Falling top, but bouncing back



(like general case on page 6 of previous note)

— Thus, <u>other</u> values of θ <u>are</u> allowed, where $\dot{\theta} = 0$, i.e., we (again) <u>expect</u> θ to <u>not</u> remain <u>constant</u>.

— However, <u>as of now</u>, we don't know whether $u(t=0)$ is at u_2 [i.e., $u(\text{at } t>0) < u(t=0)$ or θ increases] or at u_1 (θ decreases): again, gravitational expectation is it's former, but we need to show it via equations!

— First of all, $\dot{\phi}(t=0) = 0$ gives [see just below Eq. (11) of previous note] $b = a\underbrace{u(t=0)}_{\text{zero of } f}$, i.e.,

$\boxed{(b/a)}$ is <u>zero</u> of $f(u)$: let's check it.

— We have $f(b/a) = \left(1 - b^2/a^2\right)\left(\alpha - \beta\, b/a\right)$, since 2nd term in Eq. (11) for $f(u)$ vanishes for $u = b/a$.

- Now $\dot{\phi} = 0$ at $t = 0$ gives [using Eqs. (2) (4) & (5)] $b = a \cos[\theta(t=0)]$, i.e., $b = a\, u(t=0)$ [actually, same argument as before!] and (9), (10)

- On the other hand, Eqs. (3) with $\dot{\theta} = \dot{\phi} = 0$ at $t = 0$ gives $\alpha = \beta \cos[\theta(t=0)]$

- Combining above two relations shows
$\alpha/\beta = b/a$ so that indeed $f(b/a) = 0$

— Onto $\boxed{\text{showing}}$ that $\boxed{\dot{\theta}, \dot{\phi}}$ subsequently "<u>turn on</u>": let's do it <u>by contradiction</u>, i.e., suppose $\dot{\theta}, \dot{\phi}$ stay zero. ~~.~~ then

— However, $\omega_{1,2} = \dot{\phi}\, s_\theta\, {}^{s_\psi, c_\psi} \pm \dot{\theta}\, c_\psi\, {}^{, s_\psi}$ remain zero, but then Euler's equations (with torque) along $\boxed{\bar{e}_{1,2}}$ cannot be satisfied: e.g., in general, we have along $\bar{e}_1$: $I_1 \dot{\omega}_1 + \omega_2 \omega_3 (I_3 - I_2) = \tau_1$ component of torque along $\bar{e}_1$

Now, LHS = 0 (again, $\omega_{1,2}$ stay 0),
while RHS $= \underbrace{Mg\ell \sin\theta}_{\text{magnitude of torque}} \times \underbrace{\cos\psi}_{\text{component along } \bar{e}_1 \text{ (body x-axis)}}$

[Also, ψ is changing, i.e., $\neq \pi/2$ in general, since $\dot{\psi} = \omega_3 =$ constant]

Thus, we have a contradiction so that our assumption that both $\dot{\theta}$ & $\dot{\phi}$ stay 0 is wrong $\Rightarrow$ one / both of $\dot{\theta}, \dot{\phi}$ must turn on at $t > 0$.
(see page 9 for shorter / different version)

[again, as expected from figure on page 6, i.e., values of θ where $\dot{\theta} \neq 0$ are allowed.] [see Eq. (3)]

— We also have $E = \text{constant} = \underbrace{\frac{1}{2} I_3 \omega_3^2}_{\text{constant}}$

$+ \frac{1}{2} I_1 \left(\dot{\theta}^2 + \dot{\phi}^2 s_\theta^2 \right) \Big\}$ = 0 at $t = 0$, but $>$ 0 for $t > 0$

$+ Mg\ell \cos\theta \Big\}$ must then reduce (as per above argument)

i.e., $\boxed{\theta \text{ must } \boxed{\text{increase}}}$ (again, $u \equiv \cos\theta$ then drops)

$\Rightarrow$ $u(t=0)$ must then be $\boxed{u_2}$ (i.e, larger root)

So, $\boxed{\text{top}}$ falls as expected!

— what about $\boxed{\dot{\phi}}$? We have $\dot{\phi} = \frac{b - au}{1 - u^2} = 0$ at $t = 0$

[again, $u(t=0) = b/a$ is (larger) root] : as u decreases, it is clear that $\dot{\phi}(t>0) \neq 0$, i.e., top starts to also precess! [Equivalently, Eq. (2) shows that to keep p_ϕ constant (with $I_3 \omega_3$ constant) as $\cos\theta$ drops, we have to turn on 1^{st} term on RHS, i.e., $\dot{\phi}$]

— So, θ increases (u decreases) till we reach $\boxed{\text{other}}$ turning point, i.e, u_1; then top bounces back (in θ), i.e, motion in θ is bounded (again, like for central force)

— $\boxed{\text{Sanity}}$ check : at u_1, we have $\dot{\theta} = 0$ (just like at u_2, i.e, $t = 0$). Naively, intuitively, gravity should continue to make top fall (again, like we "guessed" initially for u_2), but we showed via equations above that top bounces back: this shows that "gravity makes top fall" might be too simple a reasoning!

— Then, what about repeating the argument (for θ to increase) using equations given for $u_{(2)}$ (at $t=0$), now doing it for $u_{\boxed{1}}$ at later time? However, $\dot{\phi} \neq 0$ at u_1 (again, even if $\dot{\theta} = 0$) — unlike at u_2 (at $t=0$) so that one can't really go through with the same argument

— × —

Equivalent [to page (7)] version of argument for either $\boxed{\dot{\theta} \text{ or } \dot{\phi}}$ to turn on for $\boxed{t > 0}$

— Again, assume $\dot{\theta}, \dot{\phi}$ stay 0 for all time, i.e., θ, ϕ remain at their initial values. So, $\bar{\omega} = \dot{\psi}\,\underbrace{\bar{e}_3}_{\text{body}} + \dot{\phi}\,\underbrace{\bar{e}_3}_{\text{space}} + \dot{\theta}\,\underbrace{\bar{e}_1}_{\text{line of nodes}}$

$$\rightarrow \underbrace{\dot{\psi}}_{\substack{\text{also constant} \\ (= \omega_3 \text{ for } \dot{\phi} = 0)}} \underbrace{\bar{e}_3}_{\substack{\text{fixed even in space} \\ \text{frame (since so are } \theta, \phi)}}$$

Thus, $\bar{L} = \overleftrightarrow{I} \cdot \bar{\omega} = I_3 \omega_3 \bar{e}_3$ is constant, but we do have non-zero torque here, giving a contradiction $\Rightarrow$ one/both of $\dot{\theta}, \dot{\phi} \neq 0$ at $t > 0$ (as before)

— Using more of equations, we saw at bottom of page 5 that $\ddot{u} = f'(u)/2$. So, at $t=0$, even though $\dot{u} = 0$ (either at u_1 or u_2), we see that $\ddot{u}$ (at $t=0$) $\neq 0$; thus $\dot{u} \neq 0$ at $t > 0$, i.e., $\dot{\theta}$ is turned on (θ changes) [$\boxed{\text{c.f.}}$ sleeping top case on top of page 5, where $\ddot{u}$ at $t=0$ vanishes, i.e., $\boxed{f' = 0}$ (double root)]

$f'(u_{1,2}) \neq 0$

— × —

— $\boxed{\text{Finally}}$, we can make some quantitative predictions assuming "fast" top, i.e., rotational (initial) KE is much larger than maximum possible change in PE, i.e., $\frac{1}{2} I_3 \omega_3^2 \gg 2Mgl$

— In this case, we expect effect of torque (i.e., induced nutation & precession) to be sort of "perturbation" to rotation about its own axis; in particular extent of nutation $(u_2 - u_1)$ should decrease as ω_3 increases

— Let's turn to equations: we saw earlier (top of page 7) that $\alpha = \beta\, b/a$ and $u_2 (= u \text{ at } t=0) = b/a$. Then, after some algebra, we find that

$$f(u) = (u_2 - u)\underbrace{\left[(1-u^2)\beta - a^2(u_2 - u)\right]}_{\text{contains other 2 roots of } f(u)=0}$$

— Let's look for a root in physically allowed region, i.e., the other turning point $u_1 < 1$:

$$(1-u_1^2) - \frac{a^2}{\beta}(u_2 - u_1) = 0$$

— Now, $a^2/\beta = \underbrace{I_3/I_1}_{O(1)}\ \underbrace{\frac{I_3\omega_3^2}{2Mgl}}_{\gg 1 \text{ (for fast top)}} \gg 1$, while $(1-u_1^2) \lesssim O(1)$

So, we must have $(u_2 - u_1) \ll 1$ from root equation: as an approximation, we can then set $u_1 \approx u_2$ in 1st term, giving $(u_2 - u_1) \simeq \frac{\beta \sin^2\theta_2}{a^2} \ll 1$

$$= \frac{I_1}{I_3}\ \frac{2Mgl}{I_3\omega_3^2}\sin^2\theta_2, \text{ i.e., } \frac{1}{\omega_3}$$

as anticipated, extent of nutation is small, reducing with ω_3.

— To finish the story, $\boxed{2^{nd}}$ root of above quadratic equation needs to be $\gg 1$ (i.e., unphysical): indeed, it is easy to see that it is given (approximately) by $\left(-\frac{a^2}{\beta}\right) \gg 1$, since then 1st term in root equation $\approx -u_1^2$ cancels $a^2 u_1/\beta$ from 2nd term (where we drop u_2) ... $\boxed{\text{more}}$ in GPS sec. 5.7