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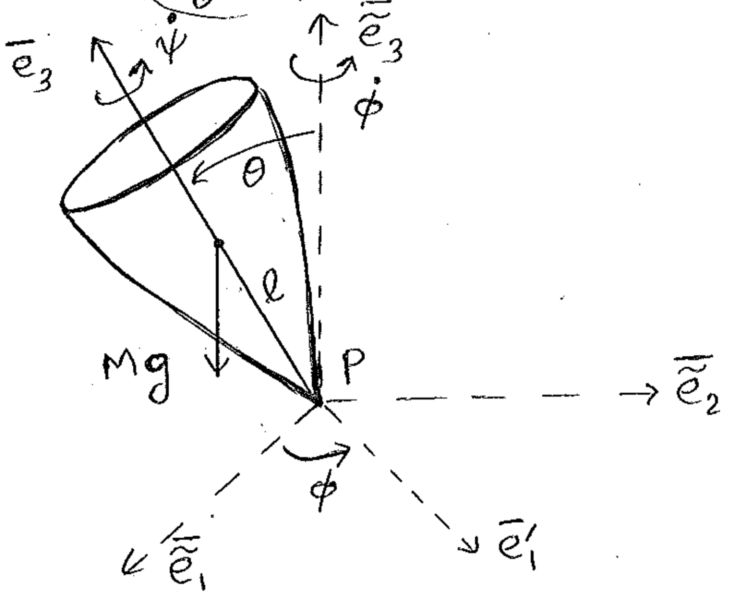
Heavy, symmetric ($I_1 = I_2 \neq I_3$) top in general

(based on section 5.7 of GPS and section 3.6 of DT)

— Symmetric top (mass M) pinned at point P moving under (vertical) gravity (COM is distance l along top, i.e., body z, axes from P



— Big picture first: unlike free top, we will find nutation (i.e., $\theta \neq$ constant or $\dot{\theta} \neq 0$) in general (as expected, i.e., top tends to "fall" under gravity)

— However, (remarkably!) we will show that uniform precession ($\dot{\phi}$ = constant) without nutation ($\dot{\theta} = 0$, θ = constant) is also possible (for certain values of $\dot{\phi}$), sort of like for free top

— Also, spinning upright can be stable or unstable (depending on how fast top is spinning) vs. for free, this would always be stable (since there is no gravity to cause instability)

— $\boxed{\text{Formalism}}$ to be used : since there <u>is</u> torque (due to gravity), overall angular momentum ($\bar{L}$) is not constant so that it is not clear if approaches (I) & (II) used for free top (see other note) are applicable here. Whereas, Euler equations [approach (III)] does not really inform us about motion of (θ, ψ, ϕ) [what we observe in space frame].

So, go back to using <u>Lagrangian</u>!

— Clearly, L = kinetic (rotational) energy + gravitational potential energy

$$= \frac{1}{2}\left[I_1\left(\omega_1^2+\omega_2^2\right)+I_3\,\omega_3^2\right] - Mgl\cos\theta \quad \leftarrow \cos\theta$$

$$= \frac{1}{2} I_1\left(\dot{\theta}^2 + s_\theta^2\,\dot{\phi}^2\right) + \frac{1}{2} I_3\left(\dot{\psi} + c_\theta\,\dot{\phi}\right)^2 - Mgl\,c_\theta \quad \ldots (0)$$

[using $\omega_1 = \dot{\phi}\, s_\theta s_\psi + \dot{\theta}\, c_\psi$; $\omega_2 = \dot{\phi}\, s_\theta c_\psi - \dot{\theta}\, s_\psi$ & $\omega_3 = \dot{\psi} + \dot{\phi}\, c_\theta$ for $\bar{\omega}$ resolved along (principal) body axes]

— Overall $\boxed{\text{strategy}}$ (I) : similarly to central force case, we will first figure out <u>constants of motion</u>; then use them to "reduce" 2nd order differential equations (i.e, Lagrange's or Newton's) to $\boxed{1^{st}}$ order (single "dots") [only $\dot{\psi}, \dot{\phi}$ appear in Eq. (0)]

— Both ψ, ϕ are cyclic coordinates (so that $p_{\psi, \phi}$ are constants

— Now, $\boxed{p_\psi} = \partial L / \partial \dot{\psi} = I_3\left(\dot{\psi} + \dot{\phi}\, c_\theta\right) = \boxed{I_3\,\omega_3} = \boxed{\text{constant}}$ ... Eq. (1)

i.e., component of $\bar{L}$ (angular momentum) along body z-axis.

— Conservation of p_ψ can then be seen from a more "elementary" argument, i.e., using $\boxed{\text{torque}}$, $\bar{\tau} = \underbrace{\bar{l}}_{\substack{\text{vector from P to}\\ \text{COM, i.e.,}\\ \text{along } \boxed{\bar{e}_3}}} \times \underbrace{M\bar{g}}_{\text{along } \boxed{\tilde{\bar{e}}_3}}$ clearly having

(about P)

$\bar{\tau}$ is along line of nodes ($\bar{e}_1'$) then

$\boxed{\text{no}}$ component along $\boxed{\bar{e}_3}$ (top z-axis) so that $L_3 = I_3 \omega_3$ = constant

— Similary, torque has zero component along $\boxed{\tilde{\bar{e}}_3}$ (i.e., space z-axis) so that $\bar{L} \cdot \tilde{\bar{e}}_3$ = constant

— Now, $\bar{L} = I\omega_1 \bar{e}_1 + I_2 \omega_2 \bar{e}_2 + I_3 \omega_3 \bar{e}_3$, with $\omega_{1,2,3}$ as given just below Eq. (0). Combining this with $\tilde{\bar{e}}_3 = \bar{e}_1 S_\theta S_\psi + S_\theta C_\psi \bar{e}_2 + C_\theta \bar{e}_3$ shows that $I_3 C_\theta (\dot{\psi} + \dot{\phi} C_\theta) + I_1 S_\theta^2 \dot{\phi} \left(\overset{\text{again}}{=} \bar{L} \cdot \tilde{\bar{e}}_3\right)$ = constant

— Above constant of motion is just $p_\phi = \partial L / \partial \dot{\phi}$, which was expected to be conserved since ϕ is cyclic. Using $\omega_3 = \dot{\psi} + \dot{\phi} C_\theta$ (= constant), we then get

$$p_\phi = I_1 S_\theta^2 \dot{\phi} + I_3 C_\theta \omega_3 = \text{constant} \quad \dots (2)$$

— Also, since L in Eq. (0) has no explicit time dependence, we get that energy is constant, i.e.,

$$E = \tfrac{1}{2} I_1 \left(\dot{\theta}^2 + \dot{\phi}^2 S_\theta^2\right) + \tfrac{1}{2} I_3 \omega_3^2 + Mgl\cos\theta = \text{constant} \quad \dots (3)$$

— We thus have $\boxed{3}$ constants of motion (p_ψ, p_ϕ & E) needed in order to carry out our plan of reducing to 1st order DE's for $\boxed{3}$ variables (θ, ψ & ϕ)

— Explicity (similarly to what was done for central force), we have 6 initial conditions for solving $\boxed{2^{nd}}$ order

DE's in 3 variables $\longrightarrow$ 3 constants of motion + 3 initial conditions for 3 1st order DE's.

— Let's $\boxed{\text{go ahead}}$ then: the idea here is to "get rid of" $\underbrace{\dot{\phi}, \dot{\psi}}$ in favor of $\boxed{\theta, \dot{\theta}}$ (using p_ψ, p_ϕ = constant) again, ϕ, ψ don't appear (explicitly).

Then, use E = constant to obtain 1st order DE for θ [again, roughly analogous to eliminating $\dot{\theta}$ in central force problem to obtain 1st order DE for r].

— In more detail, it is convenient to define (constants)

$$a = I_3 \omega_3 / I_1 \quad \cdots (4) \qquad \& \qquad b = p_\phi / I_1 \quad \cdots (5)$$

so that Eqs. (1) & (2) can be compactly (re-)written as

$$\dot{\phi} = (b - a\, c_\theta)/s_\theta^2 \quad \cdots (6) \qquad \& \qquad \dot{\psi} = \frac{I_1 a}{I_3} - \frac{(b - a c_\theta)\, c_\theta}{s_\theta^2} \cdots (7)$$

[Note that strictly speaking, these are not well-defined for $\boxed{\theta = 0}$ i.e., $s_\theta = 0$.]

Thus, as planned, Eqs. (7), (8) can be integrated to furnish $\psi(t)$, $\phi(t)$ once we plug $\theta(t)$ from step below.

— So, onto $\boxed{\text{solving for } \theta(t)}$ using 3rd / last constant of motion: we have from Eq. (3), with $\dot{\phi}$ plugged from Eq. (7)

$$E - \tfrac{1}{2} I_3 \omega_3^2 \equiv E' \,(= \text{constant}) = \underbrace{\tfrac{1}{2} I_1 \dot{\theta}^2}_{\text{"kinetic" energy "along" } \theta} + V_{\text{eff}}(\theta),$$

where $V_{\text{eff}}(\theta) = Mgl\, c_\theta$ ("original" potential energy)

$$+ \underbrace{I_1 (b - a c_\theta)^2 / (2 s_\theta^2)}_{\text{"}\phi\text{" kinetic energy}} \quad \cdots (8)$$

[Again, just like for central force, where equivalent 1D potential, $V'(r)$, combines original PE & angular KE.]

— Defining $u \equiv \cos\theta$ $(-1 \leq u \leq 1)$ and
$\alpha \equiv 2E'/I_1$ ...(9) & $\beta = 2Mgl/I_1$ ...(10),
we can (re-)write Eq. (8) as

$$\boxed{\dot{u}^2 = f(u)}, \text{ where } \boxed{f(u)} \equiv (1-u^2)(\alpha - \beta u) - (b - au)^2 \ldots (11)$$
$$= u^3 \beta + \cdots$$

is a cubic polynomial in u

[Also, Eqs. (6), (7) become $\dot{\phi} = \dfrac{b - au}{1-u^2}$ & $\dot{\psi} = \dfrac{I_1 a}{I_3} - \dfrac{u(b-au)}{1-u^2}$]

— Thus $u(t)$ can in principle be obtained (involving elliptic integrals) as $\int dt = \int du/\sqrt{f(u)}$

— Here, we will mostly content ourselves with understanding motion of top qualitatively using general features of $f(u)$ [just like $V'(r)$ for central force]

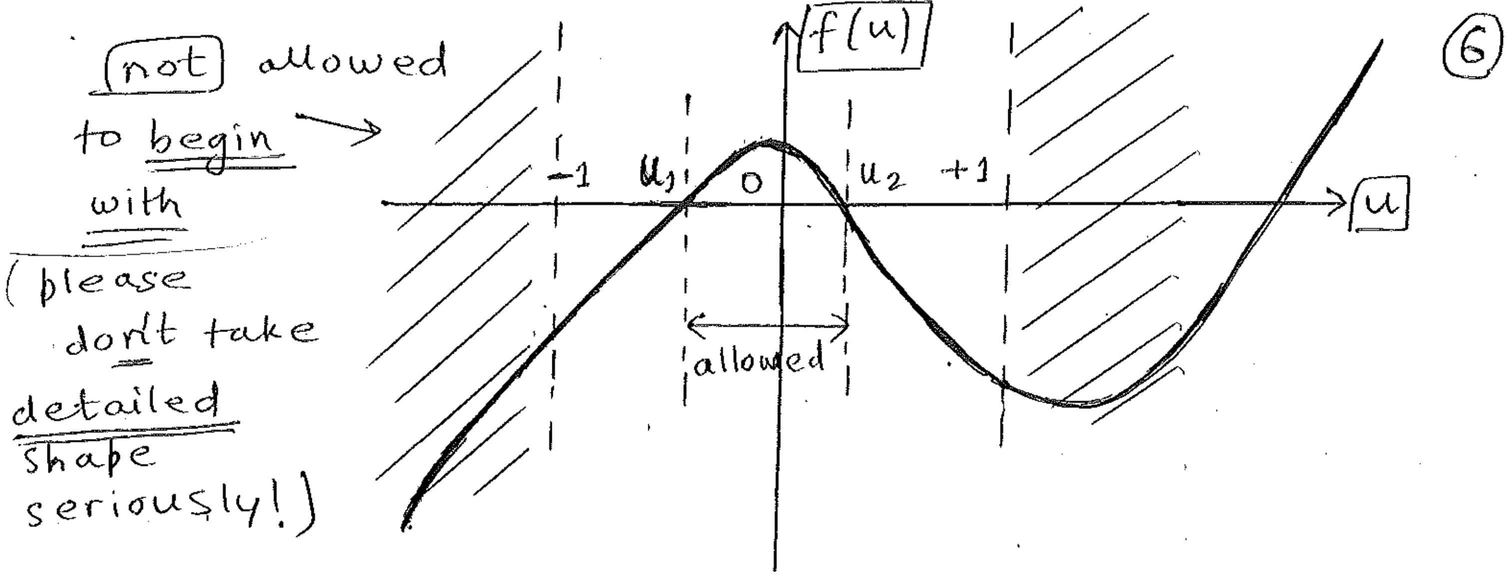
— So, onto $\boxed{\text{analysis}}$ of $\boxed{f(u)}$: clearly $f \to \pm\infty$ as $u \to \pm\infty$ [again since $f(u) = \beta \underbrace{u^3}_{\equiv \cos\theta} + \ldots$ with $\beta = \dfrac{2Mg}{I_1} > 0$]

— $\boxed{\text{Maximum}}$/physical range of u, its is $[-1, 1]$:
$f(\pm 1) = -(b \mp a)^2 \leq 0$ at its endpoints

— Also, we need $\dot{u}^2 = \boxed{f(u) \geq 0}$ over some part of $[-1, 1]$ so as to be physically relevant (again, like $\underbrace{(\tfrac{1}{2}m\dot{r}^2)}_{\text{radial KE}} \geq 0$ forces $E \geq V'(r)$ for central force)

— Finally, $f(u)$ has (at most) 3 $\boxed{\text{real}}$ zeroes (or roots of $f = 0$) over entire range of u

— Combining all above facts, we deduce the following picture of $f(u)$ (again, physically interesting cases only); this is valid in general (specific cases to be dealt with below)



— Clearly, system is restricted to be between the $\boxed{2}$ turning points $\boxed{u_{1,2}}$ [i.e., zeroes of $f(u)$ inside $(-1,1)$: again, 3^{rd} zero has to be at $u \geq 1$ given $f(u=+1) \leq 0$ and $f(u \to +\infty) \to \infty$.]

[again, analogous to boundedness of r for central force deduced simply from $V'(r)$, for given E.]