Phys 601 (by Dr. Agashe

"Summary" of Euler's equations

— First, angular momentum ($\bar{L}$) is defined (or constructed) in fixed/space/inertial frame (axes denoted by $\bar{e}_{a=1,2,3}$):

$$\bar{L} = \sum_{\text{particles}} m\left(\bar{r} \times \dot{\bar{r}}\right) \quad \ldots (1)$$

[dropping the particle index ("i") for simplicity]

— Explicitly, $\boxed{\dot{\bar{r}}}$ is velocity of particle as constructed/defined by inertial observer: here, $\bar{r}$ is position vector of particle w.r.t. fixed point, P in the body (typically the COM)

[Obviously, an observer moving with the body, i.e., fixed in the body frame, will find the particle not moving at all, i.e, velocity as constructed by this observer is zero, but that is not what enters in $\bar{L}$ above!]

— Similarly, for a free body, $\boxed{\frac{d\bar{L}}{dt} = 0}$, but we have to be careful that d/dt here is as evaluated in inertial frame (or $\dot{\bar{r}}$, i.e., velocity of particle) (looks simple, but rich structure due to tale of [2] sets of axes!)

— Once $\boxed{\bar{L}}$ is constructed as above, this (given) vector can of course be expanded in terms of (or resolved into) unit vectors in either inertial frame ($\tilde{\bar{e}}_a$) or body frame ($\bar{e}_a$):

$$\bar{L} = \begin{cases} \sum_a \tilde{L}_a(t) \tilde{\bar{e}}_a & \text{in inertial frame} \\ \sum_a L_a(t) \bar{e}_a(t) & \text{in body frame} \end{cases}$$

in general

[i.e., L_a's are components of $\bar{L}$ (again, $\bar{L}$ as defined by inertial observer), but as "seen" by observer moving with the body.]

— For a free body (i.e, $d\bar{L}/dt = 0$), in inertial frame, we get

$$\sum_a \frac{d\tilde{L}_a}{dt} \tilde{\bar{e}}_a = 0$$ [again, given that $\tilde{\bar{e}}_a$ are fixed]

$$\Rightarrow \quad d\tilde{L}_a/dt = 0$$ (i.e, separately for $\boxed{\text{each}}$ a, cf. La below...)

— However, in body frame, it's more complicated, since $d\bar{L}/dt = 0$ implies

$$\underbrace{\sum_a \frac{dL_a}{dt} \bar{e}_a}_{\text{"similar" to inertial frame}} + \sum_a L_a \underbrace{\frac{d\bar{e}_a}{dt}}_{\bar{\omega} \times (\bar{e})_a} = 0$$

"new" (vs. inertial frame): $\bar{e}_a$ are not fixed (in space frame, where d/dt is taken)

[as in Eq. 3.7 of DT, also derived in lecture]

Writing $\bar{\omega} = \omega_b \bar{e}_b$ (i.e., resolving $\bar{\omega}$ along body axes also) and using $\bar{e}_b \times \bar{e}_a = \sum_c \epsilon_{bac} \bar{e}_c$ gives

$$\sum_a \frac{dL_a}{dt} \bar{e}_a + \sum_{a,b,c} L_a \omega_b \epsilon_{bac} \bar{e}_c = 0 \quad \ldots (2)$$

Choosing $\bar{e}_1$ component of Eq. (2) gives [i.e., setting $\underline{a=1}$ in $\underline{1^{st}}$ term and $\underline{c=1}$ in $\underline{2^{nd}}$ term]

$$dL_1/dt + L_2\,\omega_3\,\epsilon_{321} + L_3\,\omega_2\,\epsilon_{231} = 0$$

$a = 2;\ b = 3$ in 2^{nd} term of Eq. 2 $\qquad a = 3,\ b = 2 \ldots$

i.e., $\boxed{dL_1/dt - L_2\omega_3 + L_3\omega_2 = 0}$ ... (3)

Similarly, resolving Eq. (2) along 2, 3 directions gives (respectively)

$$\boxed{\begin{array}{l} dL_2/dt + L_1\omega_3 - L_3\omega_1 = 0 \quad \ldots (4) \\ dL_3/dt - L_1\omega_2 + L_2\omega_1 = 0 \quad \ldots (5) \end{array}}$$

— So, in general, $\boxed{dL_1/dt \neq 0}$ (similarly for $L_{1,2}$) even for free body, as expected since it is d/dt (only) as seen by inertial-frame observer (of $\bar{L}$) that ~~is~~ zero

— Compare above discussion (of resolving vectors) to what was done earlier for $\boxed{\bar{r}}$ itself, i.e.,

$$\bar{r} = \begin{cases} \sum_a \underbrace{r_a}_{\text{constant}}\ \underbrace{\bar{e}_a(t)}_{\text{moving}}, \text{ in body-frame} \\ \sum_a \underbrace{\tilde{r}_a(t)}_{\text{varying}}\ \underbrace{\tilde{\bar{e}}_a}_{\text{constant}} \end{cases}$$

so that $\dot{\bar{r}}$ (again, d/dt here is as evaluated by inertial observer, but can be expanded in $\bar{e}_a$ or $\tilde{\bar{e}}_a$) $= \sum_a \left(\frac{d\tilde{r}_a}{dt}\right)\tilde{\bar{e}}_a = \sum_a r_a \frac{d\bar{e}_a}{dt}$

[Body-frame observer would obviously say $d\bar{e}_a/dt = 0$, i.e., velocity of particle/point in body vanishes!]

— $\boxed{\text{Why}}$ would we resolve $\bar{L}$ as above into components along body axes [given that their evolution, as in Eqs. (3)–(5) is complicated, cf. $\tilde{L}_a$ = constant (i.e., in inertial frame)] ?!

— $\boxed{\text{Because}}$ expression for L_a (whose time-evolution is desired) (cf. $\tilde{L}_a$) in terms of ω-components also along body-axes <u>and</u> body parameters/characteristics is convenient, i.e., using $\dot{\bar{r}} = \sum_a r_a \frac{d\bar{e}_a}{dt} = \sum_a r_a \, \bar{\omega} \times (\bar{e})_a = \bar{\omega} \times \underbrace{\sum r_a (\bar{e})_a}_{\bar{r}}$ in Eq. (1) [and some vector algebra] gives

$$\bar{L} = \sum m\left[|\bar{r}|^2 \bar{\omega} - (\bar{\omega} \cdot \bar{r})\, \bar{r}\right] \quad \ldots (6)$$

resolve $\bar{\omega}, \bar{r}$ along body-axes

$$= \sum_a \underbrace{\sum_b I_{ab}\, \omega_b}_{\equiv L_a} \bar{e}_a \quad \ldots (7)$$

where I_{ab} (inertia tensor) $= \sum_{\text{particles}} m\left[\delta_{ab}\left(\sum_c r_c^2\right) - r_a r_b\right]$.

(obviously, I_{ab} is time-<u>in</u>dependent, since so are r_a's)

[Just to belabor the point, we could resolve $\bar{\omega}, \bar{r}$ on RHS of Eq. (6) — thus $\bar{L}$ — along space-axes, giving $\tilde{L}_a = \sum_b \tilde{I}_{ab} \tilde{\omega}_b$, with $\tilde{I}_{ab} = \sum m\left[\delta_{ab}\left(\sum_c \tilde{r}_c^2\right) - \tilde{r}_a \tilde{r}_b\right]$. However, $\tilde{I}_{ab}$ will <u>not</u> be constant, since $\tilde{r}_a$'s vary...]

— Finally, choosing body-axes to be $\boxed{\text{principal}}$ [i.e., I_{ab} is <u>diagonal</u>] gives $L_1 = I_1 \omega_1$; $L_2 = I_2 \omega_2$ & $L_3 = I_3 \omega_3$

— Plugging above L_a's into Eqs. (3)–(5) gives $\boxed{\text{Euler's}}$ <u>equations</u>:

$$I_1 \dot{\omega}_1 + \omega_2 \omega_3 (I_3 - I_2) = 0 \; ; \; I_2 \dot{\omega}_2 + \omega_1 \omega_3 (I_1 - I_3) = 0 \; \& \; I_3 \dot{\omega}_3 + \omega_1 \omega_2 (I_2 - I_1) = 0$$

— Note that if only $\omega_1 \neq 0$ (i.e., $\omega_{2,3} = 0$) so that $\boxed{\text{rotation}}$ is only about one of principal axes, then 1st of Euler equations gives $I \dot{\omega}_1 = 0$, i.e., $\omega_1 =$ constant. Also, $L_1 = I_1 \omega_1 \neq 0$, whereas $L_{2,3} = I_{2,3}\, \omega_{2,3} = 0$ so that $\bar{L} \propto \bar{\omega}$ (i.e., both are in same direction) : this is the simple rotation (discussed also earlier in context of $\bar{L}$)

— In general, $\omega_{1,2,3}$ are all non-zero, i.e., rotation is not about only one principal axes. In this case 1st of Euler equations gives $\dot{\omega}_1 \neq 0$ (since 2nd/3rd terms are non-zero), i.e., ω_1 (hence $L_1 = I_1 \omega_1$) $\neq$ constant even though we have free rigid body, i.e., $d\bar{L}/dt = 0$

— In short, Euler equations (thought of as determining evolution in time of $\omega_{1,2,3}$) are coupled, non-linear, but with constant coefficients

— Alternatively, we can consider time evolution of $\tilde{\omega}_{1,2,3}$ (i.e., components of $\bar{\omega}$ along space/fixed axes) : plugging $\tilde{L}_a = \sum_b \tilde{I}_{ab}(t)\, \tilde{\omega}_b$ into $d\tilde{L}_a/dt = 0$ gives

$$\boxed{\sum_b d\tilde{I}_{ab}/dt\, \tilde{\omega}_b + \sum \tilde{I}_{ab}\, \dot{\tilde{\omega}}_b = 0}$$

i.e., coupled, linear, but with time-dependent coefficients.