Here, we solve for the *complete* motion of simple harmonic oscillator using the Hamilton-Jacobi (H-J) method (based on GPS section 10.2). The Hamiltonian is

$$H(a, p) \;=\; \frac{p^2}{2m} + \frac{1}{2} m\omega^2 q^2 \tag{1}$$

where the force constant has been expressed in terms of the angular frequency (as usual):

$$\omega \;=\; \sqrt{\frac{k}{m}} \tag{2}$$

Since H is time-indepednent, we can use Hamilton's characteristic function, $W(q, \alpha)$, which satisfies the H-J equation:

$$H\left(q, \frac{\partial W}{\partial q}\right) \;=\; \alpha \tag{3}$$

i.e., in this case, we have

$$\frac{1}{2m}\left[\left(\frac{\partial W}{\partial q}\right)^2 + m^2\omega^2 q^2\right] \;=\; \alpha \tag{4}$$

whose formal solution is

$$W(q, \alpha) \;=\; \sqrt{2m\alpha}\int^q dq' \sqrt{1 - \frac{m\omega^2 q'^{\,2}}{2\alpha}} \tag{5}$$

with Hamilton's principal function being given by

$$S(q, \alpha, t) \;=\; W(q, \alpha) - \alpha t \tag{6}$$

The "old" momentum is then given by

$$p \;=\; \frac{\partial S}{\partial q} = \frac{\partial W}{\partial q} \tag{7}$$

$$\;=\; \sqrt{2m\alpha}\sqrt{1 - \frac{m\omega^2 q^2}{2\alpha}} \tag{8}$$

Evaluating the above at initial time ($t = 0$), when $q = q_0$ and $p = p_0$, we get

$$p_0 \;=\; \sqrt{2m\alpha}\sqrt{1 - \frac{m\omega^2 q_0^2}{2\alpha}} \tag{9}$$

or the *new* (constant along the path) momentum in terms of initial conditions:

$$\alpha\left(q_0, p_0\right) \;=\; \frac{p_0^2}{2m} + \frac{1}{2} m\omega^2 q_0^2 \tag{10}$$

i.e., as expected it is simply the (constant) energy, E.

Moving onto the new (constant along the path) *coordinate*, we use

$$\frac{\beta}{\omega} = \frac{\partial S}{\partial \alpha} \tag{11}$$

$$= -t + \frac{\partial W}{\partial \alpha} \tag{12}$$

The second term in above can be evaluated as follows [i.e., derivative hits α outside *and* inside integral in Eq. (5)]:

$$\frac{\partial W}{\partial \alpha} = \sqrt{2m} \int^q dq' \left(\frac{\sqrt{1 - \frac{m\omega^2 q'^{\,2}}{2\alpha}}}{2\sqrt{\alpha}} + \sqrt{\alpha} \frac{\frac{m\omega^2 q'^{\,2}}{2\alpha^2}}{2\sqrt{1 - \frac{m\omega^2 q'^{\,2}}{2\alpha}}} \right) \tag{13}$$

$$= \frac{\sqrt{2m}}{2\sqrt{\alpha}} \int^q dq' \frac{1}{\sqrt{1 - \frac{m\omega^2 q'^{\,2}}{2\alpha}}} \tag{14}$$

$$= \frac{\sqrt{2m}}{2\sqrt{\alpha}} \left[\frac{\arcsin \left(q\sqrt{\frac{m\omega^2}{2\alpha}} \right)}{\sqrt{\frac{m\omega^2}{2\alpha}}} \right] \tag{15}$$

(where constant of integration chosen to be 0, without loss of generality, since it can be absorbed into constant β anyway) so that

$$\frac{\beta}{\omega} = -t + \frac{1}{\omega} \arcsin \left(q\sqrt{\frac{m\omega^2}{2\alpha}} \right) \tag{16}$$

Setting further $t = 0$ in above, we get

$$\beta = \arcsin \left(q_0\sqrt{\frac{m\omega^2}{2\alpha}} \right) \tag{17}$$

Plugging α from Eq. (10) into above, we find (after some algebra) the new (constant) coordinate in terms of initial conditions:

$$\tan \beta \left(q_0, p_0 \right) = \frac{m\omega q_0}{p_0} \tag{18}$$

Inverting Eq. (16) (for *general* time), we have solved the problem, i.e.,

$$q(t) = \sqrt{\frac{2\alpha}{m\omega^2}} \sin \left(\omega t + \beta \right) \tag{19}$$

with $p(t)$ given by plugging above $q(t)$ into Eq. (8):

$$p(t) = \sqrt{2m\alpha} \cos \left(\omega t + \beta \right) \tag{20}$$

where α, β are given in terms of initial conditions as in Eqs. (10) and (17).

Thus, we see that Hamilton's *principal* function:

$$S(q,\alpha,t) \;=\; W(q,\alpha)-\alpha t \tag{21}$$

is the generator a canonical transformation which (for the case of simple harmonic oscillator) takes us to a new coordinate which is simply the phase *constant* and with energy as the new (also constant) momentum.

Even though it is not really needed, in general, (for example, *away* from the path) we have the (full) canonical transformation given by

$$P(q,p)\ \text{(which is } \alpha \text{ along path)} \;=\; \frac{p^2}{2m}+\frac{1}{2}m\omega^2 q^2 \tag{22}$$

which simply follows from Eq. (8) by $\alpha \to P$ (note that the new momentum is a function only of old variables, i.e., *not* explicitly involving time) and plugging above P as α into Eq.(16), we get (after some algebra) the new coordinate (again, with $\beta/\omega \to Q$)

$$Q(q,p)\ \left(\text{which is } \frac{\beta}{\omega} \text{ along path}\right) \;=\; -t+\arctan\frac{m\omega q}{p} \tag{23}$$

i.e., *with* explicit time-dependence.