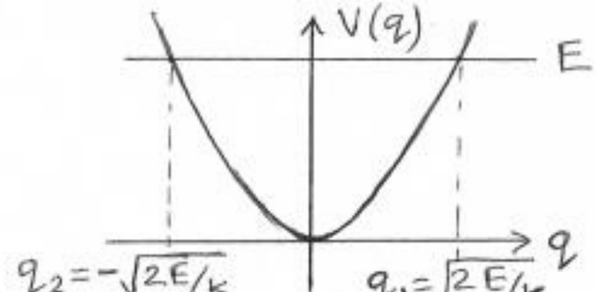

Here, we solve for the simple harmonic oscillator (SHO) frequency (only) using action-angle variables, based on last part of GPS section 10.6. The Hamiltonian is given by:

$$H(q,p) \;=\; \frac{p^2}{2m}+\frac{k}{2}q^2 \tag{1}$$

Thus, the action variable is

$$I(E) \;=\; \frac{1}{2\pi}\oint p\,(q;E)\,dq \tag{2}$$

with $p\,(q;E)$ obtained from Eq. (1) by setting $H=E$. So, we get

$$I(E) \;=\; \frac{1}{2\pi}\oint\sqrt{2m\left(E-\frac{k}{2}q^2\right)} \tag{3}$$

The turning points (i.e., $p=0$) are given by (see figure above):

$$q_{1,2}=\pm\sqrt{\frac{2E}{k}} \tag{4}$$

so that the integral in Eq. (3) can be simplified using the following substitution (α is the new variable):

$$\begin{aligned} q &= \sqrt{\frac{2E}{k}}\sin\alpha \\ dq &= \cos\alpha\, d\alpha\sqrt{\frac{2E}{k}} \end{aligned} \tag{5}$$

i.e., $\alpha=\pi/2, 3\pi/2$ correspond to the 2 turning points in Eq. (4), while $\alpha=0,\pi$ and 2π are midpoints of the motion. Plugging Eq. (5) into Eq. (3) gives (with full cycle being $\alpha=0$ to 2π)

$$\begin{aligned} I(E) &= \frac{E}{\pi}\sqrt{\frac{m}{k}}\int_0^{2\pi}\cos^2\alpha\, d\alpha \\ &= E\sqrt{\frac{m}{k}} \end{aligned} \tag{6}$$

so that

$$\begin{aligned} \omega &= \frac{1}{\frac{\partial I(E)}{\partial E}} \\ &= \sqrt{\frac{k}{m}} \end{aligned} \tag{7}$$

as expected.