"Summary" of Hamilton-Jacobi (HJ) method (Sec. $\boxed{10.1}$ of GPS)

$\boxed{1}$

- General idea: $\boxed{\text{find}}$ (from q, p) CT to go to new variables (Q, P) which are both constants (vs. only new momentum being constant for AA variables): $q, p(Q, P, \textcircled{t})$ or $Q, P(q, p, \textcircled{t})$ (known functions)

  crucial (see below)

- Evaluate $Q, P(q, p, t)$ at t_0 to get the constants (denoted by β, α) in terms of t_0, q_0, p_0
- Plug $\alpha, \beta(q_0, p_0, t_0)$ as P, Q into $q, p(Q, P, t)$ to get $q, p\left[\alpha(q_0, p_0, t_0), \beta(q_0, p_0, t_0), t\right]$
- Onto details: Q, P constant $\left(\dot{Q} = \partial K/\partial P = 0,\ \dot{P} = -\partial K/\partial Q = 0\right)$
  $\Rightarrow$ new/transformed Hamiltonian $K = 0$; for which (earlier) $Q, P(q, p, \boxed{\text{no}}\ t)$ won't do because that gives $K(Q, P) = \widetilde{H}(Q, P)$ of before, i.e., $H\left[q(Q, P), p(Q, P)\right] \neq 0$
- So, need to develop more general (time-dependent) theory of CT (a la GPS sec. 9.1 or sec. 4.4.3 of DT)

$$p = \frac{\partial F_2}{\partial q}(q, P, t)$$, inverting which gives $P(q, p, t)$

& $$Q = \frac{\partial F_2}{\partial P}(q, P, t)$$, where plugging above $P(q, p, t)$ gives $Q(q, p, t)$

(F_2 is called generating function)

- Then, $K(Q, P, t) = H(\underbrace{q, p}, t) + \partial F_2/\partial t\ (q, P, t)$

  write in terms of Q, P

P.T.O.

— Above is general CT, while here we want to set $K = 0 \Rightarrow F_2$ (denoted now by S) satisfies H-J equation:

$$\boxed{H\left(q, \underbrace{\partial S/\partial q}_{p}, t\right) + \frac{\partial S}{\partial t} = 0}$$

— Step $\boxed{I}$ in this method is then to solve above to get $S(\underset{1 \ldots n}{q_i}; \underbrace{\alpha_1 \ldots \alpha_n}_{\text{constants}}; t)$ (n+1 variables in PDE, but 1 is irrelevant since S & S + α are both solutions)

$\Rightarrow$ can take $\alpha_1 \ldots \alpha_n$ to be $P_1 \ldots P_n$ (new constant momenta)

[again, both sets of quantities can be written in terms of initial conditions]

— Step $\boxed{II}$: Use $p = \partial S / \partial q$ $\boxed{\text{at}}$ t_0 to get $\alpha(q_0, p_0, t_0)$ ← known from step I

— Step $\boxed{III}$: Use $Q(= \underset{\text{unknown}}{\beta}) = \partial S(q, \underset{\text{known}}{\alpha}, t)/\partial \alpha (= P)$ at $\boxed{t_0}$ to get $\beta(q_0, p_0, t_0)$

— Step $\boxed{IV}$: Invert $\beta = \partial S(q, \alpha, t)/\partial \alpha$ (with α, β now known) to get $q[\alpha(q_0, p_0, t_0), \beta(q_0, p_0, t_0), t]$ ← general now

— Step $\boxed{V}$: Plug above q into $p = \partial S(q, \alpha, t)/\partial q$ (again, do $\partial/\partial q$ first) to get $p(\alpha, \beta, t)$