Here we will show (following section 49 of chapter 7 from Landau, Lifshitz) that action variable (for 1D system with period T), i.e.,

$$I(E;\lambda) \;=\; \frac{1}{2\pi}\int_0^T \left(\text{denoted by } \oint\right) p(q;E,\lambda)\,dq \tag{1}$$

where $p(q;E,\lambda)$ is obtained by solving

$$H(q,p;\lambda) \;=\; E \tag{2}$$

is *adiabatically* invariant, i.e.,

$$\frac{dI(E;\lambda)}{dt} = 0 \text{ at } O(\dot{\lambda} \text{ or } \epsilon) \tag{3}$$

where

$$\begin{aligned}\epsilon \;&\equiv\; \frac{\bar{\dot{\lambda}}T}{\bar{\lambda}} \\ &\ll\; 1 \text{ (i.e., slowly varying) } \lambda\end{aligned} \tag{4}$$

with "dot" standing for time-derivative (as usual) and "bar" denoting average over one period (T). Similarly, $\dot{\lambda}$ itself is assumed to be slowly varying, i.e., $\bar{\ddot{\lambda}}T/\bar{\dot{\lambda}} \ll 1$ [for simplicity, we can take this factor to be $O(\epsilon)$ also)].

The above behavior of I (to be proven below) is to be compared to

$$\dot{E} \propto \dot{\lambda} \text{ (or } \epsilon) \tag{5}$$

i.e., if the Hamiltonian (via its parameter λ) is time-*dependent*, then the energy of the system is not constant, that too at *leading* order in the rate of change of that parameter (cf. I above being constant at *this* order). However, given that $\epsilon \ll 1$, the *relative* change in E over T is *small*. [Of course, q and p change by $O(1)$ over a time of order T, *even if* λ is constant.]

From Eq. (2), we have (along the actual path)

$$\frac{dE}{dt} \;=\; \frac{\partial H}{\partial \lambda}\dot{\lambda} \tag{6}$$

where the 1st factor, i.e., $\partial H/\partial\lambda$ is rapidly evolving over T (since it contains the q and p), while 2nd factor ($\dot{\lambda}$) is slowly varying. So, in taking average over T, we can (approximately) pull $\dot{\lambda}$ outside, i.e., neglecting (ϵ^2) effects, we have

$$\overline{\frac{dE}{dt}} \;\approx\; \overline{\frac{\partial H}{\partial \lambda}}\dot{\lambda} \tag{7}$$

where "$\dot{\lambda}$" can be thought of as average over T, but since its *instantaneous* value differs from the average at higher order in ϵ, we will keep using it with*out* a "bar". Note that within our approximations, i.e., keep only $O(\epsilon)$ effects, λ is taken to be *constant* while averaging $\partial H/\partial\lambda$ above, i.e., $\partial H/\partial\lambda$ is varying (only) due to q and p (this is the change with time that is being averaged).

Explicitly, we get

$$\overline{\frac{\partial H}{\partial \lambda}} = \frac{1}{T} \oint \frac{\partial H}{\partial \lambda} dt \tag{8}$$

We can use $\dot{q} = \partial H / \partial p$ or

$$dt = \frac{dq}{\frac{\partial H}{\partial p}} \tag{9}$$

thus

$$\begin{aligned} T &= \oint dt \\ &= \oint \frac{dq}{\frac{\partial H}{\partial p}} \end{aligned} \tag{10}$$

in Eq. (8) so that Eq. (7) then gives

$$\overline{\frac{dE}{dt}} \approx \dot{\lambda} \frac{\oint \frac{\partial H}{\partial \lambda} \frac{dq}{\frac{\partial H}{\partial p}}}{\oint \frac{dq}{\frac{\partial H}{\partial p}}} \tag{11}$$

Again, integral is over actual path for given, *constant* λ so that $H = E$ (constant), i.e.,

$$\begin{aligned} \frac{dH}{d\lambda} &= \frac{\partial H}{\partial \lambda} + \frac{\partial H}{\partial p} \frac{\partial p}{\partial \lambda} \\ &= 0 \end{aligned} \tag{12}$$

where 2nd term in 1st line above follows from $p(q; E, \lambda)$ as found using Eq. (2): just to be clear, note that q is instead taken to be *in*dependent variable (so there is *no* "$\partial H / \partial q$" in above).

Plugging $\partial H / \partial \lambda$ from Eq. (12) into Eq. (11) – and massaging a bit – gives

$$\overline{\frac{dE}{dt}} \approx -\frac{d\lambda}{dt} \frac{\oint dq \frac{\partial p}{\partial \lambda}}{\oint dq \frac{\partial p}{\partial E \text{ (or } H)}} \tag{13}$$

which can be re-written as

$$\oint dq \left(\frac{\partial p}{\partial E} \frac{dE}{dt} + \frac{\partial p}{\partial \lambda} \frac{d\lambda}{dt} \right) \approx 0 \tag{14}$$

(where $\dot{\lambda}$ and $\overline{dE/dt}$ can be taken inside integral within our approximations) or

$$\frac{d}{dt} \oint \left[p(q; E, \lambda) dq \right] \approx 0 \tag{15}$$

i.e., action variable of Eq. (1) is a constant (again, at leading order in ϵ).