"Summary" of action-angle (AA) variables

(for 1D a periodic (system / motion)

— AA is a specific canonical transformation (CT) with following features

-(I) New coordinate θ is cyclic [why it's called θ will be clear in (II) below] so that new momentum $\boxed{I \text{ is constant along path}}$, i.e.,

$H(q,p) = H[q(\theta, I), p(\theta, I)] \equiv \tilde{H}(\theta, I)$ in general, → transformed Hamiltonian

but here $\tilde{H}$ (I only, no θ)

so that $\dot{I} = -\partial\tilde{H}(\theta, I)/\partial\theta$ (in general)

$= -\partial\tilde{H}(I)/\partial\theta = 0$ here

Since along path, $H(q,p)$ or $\tilde{H}(I)$ is also constant (E) we can write (along path) $\boxed{I(E)}$

so, $\dot{\theta} = \partial\tilde{H}(\theta, I)/\partial I$ (in general) $= \partial\tilde{H}(I)/\partial I$ here

gives $\theta(t) = c(E)\,t + \text{constant}$, where $c(E) \equiv \dfrac{1}{\partial I(E)/\partial E}$

i.e., θ is $\boxed{\text{not}}$ constant (even if simply linear in time)

-(II) In addition, if we choose

$$\boxed{I(E) = \frac{1}{2\pi}\oint_{\text{cycle}} p(q;E)\,dq}$$

(see note posted), where $p(q;E)$ is obtained from $\boxed{H(q,p) = E}$

we can show that $c(E)\left(= 1/[\partial I(E)/\partial E]\right) = \omega(E) = \dfrac{2\pi}{T(E)}$, i.e., angular frequency, so that θ goes thru 2π in time T, thus deserves to be called "angle".

P.T.O.

Action variable is $\boxed{\text{adiabatic invariant}}$

- Suppose one of parameters in Hamiltonian changes with time, but slowly, i.e., $H\,[q, p, \lambda(t)]$, with $\Delta\lambda$ in time $T \ll \lambda(\text{average})$, i.e., $\dot{\lambda}T/\lambda \equiv \epsilon \ll 1$

· [clearly q, p themselves change by $O(1)$ during time T] since H is time-dependent now

- We then expect $\dot{E} \propto \dot{\lambda}$, i.e., E changes at $O(\epsilon)$
- So, we might guess that a combination of E & λ does not change at $O(\epsilon)$
- Indeed, we can show that action is that combination, i.e., $\dot{I} \propto (\dot{\lambda})^{②}$ or larger

[For this discussion, $I(E)$ above should perhaps be written as

$I(E;\lambda)$, where dependence on λ comes from $H(\underbrace{q, p}_{\text{variables}}; \lambda) = E$. $\hookrightarrow$ parameter

In this way, we see that change with time in I due to above (explicit) dependence on λ (which is varying) "cancels" that due to E itself changing (again $\propto \dot{\lambda}$)]