Phys 601 (Dr. Agashe)

Relating $\boxed{\text{scattering angle}}$ of particle 1 in laboratory (lab) frame (θ_{lab}) to that in center-of-mass (COM) frame (θ_{COM}) [based on GPS, sec. 3.11]

[Recall that $\boxed{\theta_{COM} = \theta}$ of effective 1-particle scattering, i.e., with fixed center of force, as in GPS sec. 3.10 and previous lectures. Thus above relation will connect also $\boxed{\theta_{lab}}$ to $\boxed{\theta}$ so that we can transfer analysis done with θ, e.g., computation of cross-section (σ) to lab frame.]

Outline of procedure

- Basic idea is to use "velocity triangle" (i.e., relate velocities of particle 1 in COM & lab frames): after all, θ's are also directions of velocities
  - first, obtain relation between $\cos\theta_{COM}$ and $\cos\theta_{lab}$ in terms of $\boxed{v_0/v_1'}$, where v_0 is initial speed of particle 1 in lab frame and v_1' is its final speed in COM frame
  - then use $\boxed{v_0/v}$, where v is final relative speed of 2 particles
  - why above step? Because v_0 (initial relative speed) $/v$ can be conveniently expressed in terms of (kinetic) energy loss factor

called "Q-value" of process (where elastic collision, i.e., no kinetic energy lost, corresponds to $Q = 0$)
$\boxed{\text{see Eq. (6)}}$ on page 3

— Use above $\overbrace{\theta_{COM}\ (=\theta) - \theta_{lab} \text{ relation}}$ to convert $\sigma(\theta)$ [obtained earlier] into $\boxed{\sigma'(\theta_{lab})}$, i.e., what's actually measured : $\boxed{\text{see Eq. (12)}}$ on page 5

— simple, specific cases / checks

— consider energy transferred by particle 1 to 2 (latter being at rest initially in lab frame)

Onto details!

— Notation : $\boxed{\bar{r}_1, \bar{v}_1}$ for particle 1 after scattering in lab frame
(position vector, velocity)

$\boxed{\bar{r}_1', \bar{v}_1'}$ for particle 1 after scattering in COM frame

$\boxed{\bar{R}, \bar{V}}$ for COM in lab frame (of course, $\bar{V}$ is constant) particle 2 in lab frame

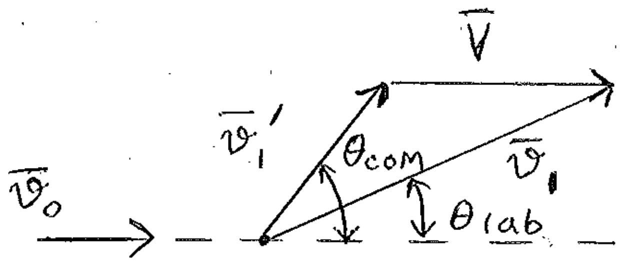
— Using $\bar{R} = (\bar{r}_1 m_1 + \bar{r}_2 m_2)/(m_1+m_2)$ and $\bar{r} = \bar{r}_2 - \bar{r}_1$, we have

$$\bar{r}_1' = -m_2 \bar{r}/(m_1+m_2) \text{ so that } \bar{v}_1' = (\mu/m_1)\, \bar{v} \quad \ldots (1)$$

where $\bar{v}$ is relative velocity ($\bar{r}$ is relative position vector) after collision and $\mu = m_1 m_2/(m_1+m_2)$ is the reduced mass.

— Basic Velocity triangle : clearly $\underset{\text{in lab frame}}{\bar{r}_1} = \underset{\text{of COM in lab}}{\bar{R}} + \underset{\text{in COM}}{\bar{r}_1'}$ so that $\bar{v}_1 = \bar{V} + \bar{v}_1' \quad \ldots (2)$

or in picture form :



— Relating $\bar{V}$ to $\bar{v}_0$: assuming particle 2 is at rest initially in lab frame, conservation of (linear) momentum in lab frame gives

$$m_1 \bar{v}_0 \text{ (initial)} = m_1 \bar{v}_1 + m_2 \bar{v}_2 \text{ (final)}$$

$$= m_1 (\bar{V} + \bar{v}_1') + m_2 (\bar{V} + \bar{v}_2')$$

2nd particle in COM frame

$$= (m_1 + m_2)\bar{V} + \underbrace{(m_1 \bar{v}_1' + m_2 \bar{v}_2')}$$

total momentum (after collision) in COM frame = 0 (by definition of COM)

i.e, $\bar{V} = m_1 \bar{v}_0 / (m_1 + m_2)$

$= (\mu / m_2) v_0$ ... (3)

Obviously, [$\bar{V}$ is in same direction as $\bar{v}_0$, as already used in above picture]

— $\boxed{\text{Back}}$ to velocity triangle : matching horizontal components of Eq. (2) [or in picture] gives

$$v_1 \cos\theta_{lab} = v_1' \cos\theta_{COM} + V \quad \ldots (4)$$

whereas considering lengths of sides of triangle gives

$$v_1^2 = (v_1' \cos\theta_{COM} + V)^2 + (v_1 \sin\theta_{COM})^2$$

$$= v_1'^2 + V^2 + 2 v_1' V \cos\theta_{COM} \quad \ldots (5)$$

— Using magnitude of Eq. (3) in Eqs. (4), (5) gives 1st relation between 2 θ's :

$$\cos\theta_{lab} = (\cos\theta_{COM} + \rho) \Big/ \sqrt{1 + \rho^2 + 2\rho\cos\theta_{COM}} \quad \ldots (6)$$

where $\rho \equiv \frac{\mu}{m_2} \underbrace{\frac{v_0}{v_1'}}$ ... (7)

speed after collision in COM frame

— However, v_1' appearing in ratio of speeds in Eq. (7) is not yet known (while v_0 is)

— So, we trade v_1' for v using Eq. (1) to give

$$\rho = (m_1/m_2)(v_0/v) \quad \ldots (8)$$

— Well, $\boxed{v}$ (relative speed after collision) is also not yet known, but can be obtained using conservation of energy: in effective 1-body picture, (assuming original collision is elastic) it is clear that speed of particle is same (well) before and after collision (since potential energy = 0 in both these positions so that we can set kinetic energies to be equal), i.e.,

$$\underset{\text{relative speed before collision}}{v_0} = v \quad (\text{elastic}) \ \ldots (9)$$

(used in above argument)

[Recall effective $\boxed{1}$-particle velocity (/position is really the relative velocity/ position of 2 particles, which is how $\bar{v}, \bar{v}_0$ "appeared" initially in any inertial frame.]

— In general, i.e., if some (total) kinetic energy is lost in collision, we define "Q-value" (energy-loss factor) of collision by

$$\underbrace{\mu v^2/2}_{\text{final KE of effective 1-body}} = \underbrace{\mu v_0^2/2}_{\text{initial...}} + Q \ \ldots (10)$$

so that [using Eq. (10) in Eq. (8) + algebra]

$$\rho = (m_1/m_2) \Big/ \sqrt{1 + \frac{m_1+m_2}{m_2}\frac{Q}{E}} \quad \ldots (11)$$

where $E = \frac{1}{2} m v_0^2$ is initial energy of particle 1 in lab frame

Onto $\boxed{\text{cross-sections}}$

— Connecting $\sigma[\theta_{\text{COM}} (= \theta)]$ to $\sigma'(\theta_{lab})$ involves a sort of Jacobian of $\theta_{(\text{COM})} \to \theta_{lab}$ "transformation", i.e., equating number of particles in given solid angle element is same, whether expressed in terms of θ_{lab} or θ :

$$2\pi I \, \sigma(\theta) \sin\theta \, d\theta = 2\pi I \, \sigma'(\theta_{lab}) \sin\theta_{lab} \, d\theta_{lab}$$

so that using Eq. (6) (+ algebra) gives

$$\sigma'(\theta_{lab}) = \sigma(\theta) \left(1 + \rho^2 + 2\rho\cos\theta\right)^{3/2} / \left(1 + \rho\cos\theta\right) \quad \ldots (12)$$

[Again, both σ's are measured in lab frame, but just written in terms of different θ's.]

— Onto $\boxed{\text{specific}}$ cases

(a) $\boxed{\text{elastic}}$ collision and $\boxed{m_1 = m_2}$ so that $\rho = 1$ [as per Eqs. (8), (9)], giving [using Eq. (6)]

$$\cos\theta_{lab} = \sqrt{\frac{1 + \cos\theta}{2}} = \cos\frac{\theta}{2} \Rightarrow \boxed{\theta_{lab} = \theta/2}$$

i.e., scattering angle in lab frame $\leq \pi/2$ (since $\theta \leq \pi$) [only in forward hemisphere]

and $\sigma'(\theta_{lab}) = 4\cos\theta_{lab}\,\sigma(\theta)$ so that isotropic scattering in θ does not imply same in lab frame

(b) $\boxed{m_2 \gg m_1}$ and elastic gives $\rho \simeq 0$, i.e. $\boxed{\theta_{lab} = \theta}$ as expected : in lab frame, recoil of 2nd (very heavy) particle (at rest initially) is then negligible so that lab frame $\approx$ effective 1-body picture.

— Finally, even for elastic collision (again, total KE is constant), 1st particle $\boxed{\text{transfers}}$ part of its intial $\boxed{\text{energy}}$ to 2nd (target) particle : this degree of slowing down is given by

$$\frac{v_1^2 \leftarrow \text{final speed in lab frame}}{v_0^2 \leftarrow \text{initial...}} = \left(\frac{\mu}{m_2 \rho}\right)^2 \left(1 + \rho^2 + 2\rho\cos\theta\right) \quad \left(\underline{\text{in general}}\right) \quad \ldots (13)$$

by plugging v_1' in terms of ρ, v_0 [from Eq. (7)] and V in terms of v_0 [from Eq. (3)] into RHS of Eq. (5)

— for <u>elastic</u> collision ($\rho = m_1/m_2$ from Eqs. (8), (9)), we get

$$E_1/E_0 = \left(1 + 2\rho\cos\theta + \rho^2\right) / (1+\rho)^2 \quad (\text{elastic})$$

(ratio of energies of particle 1 before / after collision)

use $\theta_{lab} = \theta/2$

$$\rightarrow \frac{(1 + \cos\theta)}{2} = \cos^2\theta_{lab} \text{ for } \boxed{m_1 = m_2}$$

i.e., for maximum scattering angle $(\theta_{lab} = \pi/2)$, <u>all</u> of energy of incident particle is transferred to 2nd (of same mass), i.e., incident particle is brought to rest