Phys 601 by Dr. Agashe

Notes on free, symmetrical top ($I_1 = I_2 \neq I_3$) : 3 different approaches giving same results (based on Landau, Lifshitz (LL) § 33, 35 & 36)

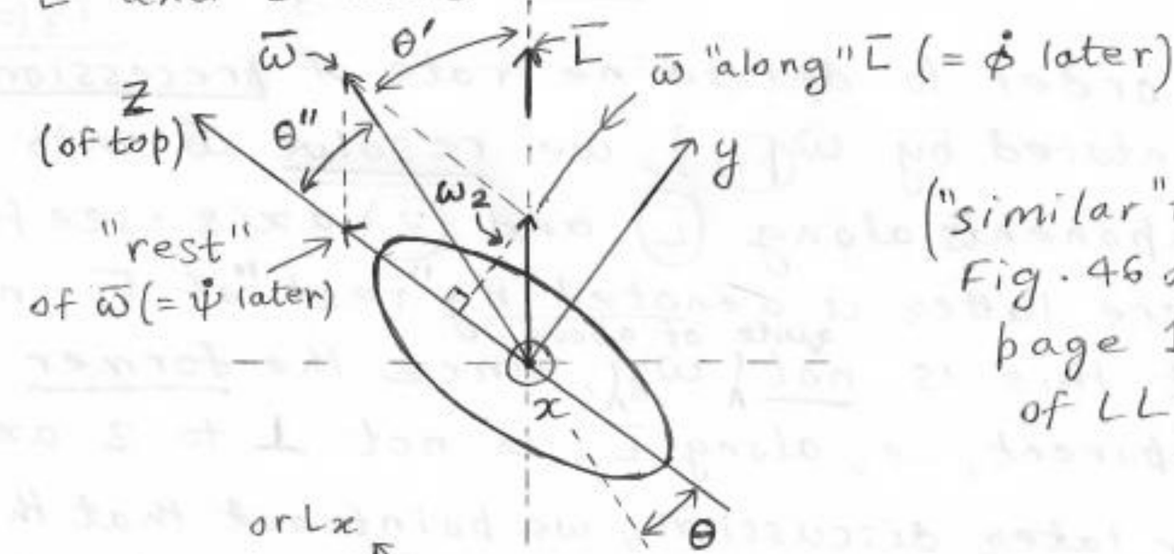
(or $I_{xx}$, or $I_{yy}$, or $I_{zz}$)

## (I). Using law of conservation of angular momentum

— $\boxed{\bar{L}}$ = constant : choose it to be vertical

— $\boxed{z}$ (symmetry) axis of top (henceforth, often called simply "axis" of top) and $\bar{L}$ in plane of paper, but at an angle $\theta$ relative to each other

— Using symmetry about top axis, $\boxed{\text{choose}}$ principal axes $\boxed{x \,\&\, y}$ so that, at (any) given instant, $\boxed{x}$-axis is perpendicular to plane of $\bar{L}$ and $z$-axis




("similar" to Fig. 46 on page 107 of LL)

— Thus, we have $L_1 = 0 = I_1 \omega_1 \Rightarrow \omega_1 = 0$ (or $L_x$)

so that $\boxed{\bar{\omega}}$ is in same plane as $\bar{L}$ and $z$-axis of top (again, latter axis is that of "usual" rotation or "spin" of top)

— In turn, velocity of any point on z-axis ②
of top (given by $\bar{\omega} \times \bar{r}$ as usual) is
out of this plane, i.e., axis of top rotates
about direction of $\bar{L}$ (called precession),
again in addition to top itself rotating
about its own (z-) axis $\Rightarrow$ $\bar{\omega}$ also rotates about $\boxed{\bar{L}}$ with $\omega_{pr.}$ since $\bar{\omega}$, $\bar{L}$, z-axis in 1 plane
— Onto formulae: had $\dot{\theta}$ been non-zero,
a point on axis of top would have a velocity
component in plane of figure. However,
we argued above (based on $\omega_1 = 0$ or $\bar{\omega}$ being
in plane) that velocity of axis is purely out
of plane $\Rightarrow$ $\dot{\theta}$ must vanish, i.e., $\boxed{\theta = \text{constant}}$ ...(1)
— $\boxed{\text{Spin}}$ of top is just component of $\bar{\omega}$ along
its (z-) axis, i.e., $\boxed{\omega_3} = L_3/I_3 = \dfrac{L \cos\theta}{I_3} \ldots(2)$ (= constant)
— In order to determine rate of precession
(dentoted by $\omega_{pr}$), we resolve $\bar{\omega}$ into
components along $\bar{L}$ and z-axis : see figure
(where latter is denoted by "rest" of $\bar{\omega}$ : note
that this is not quite $\omega_3$ of above, since the former
component, i.e., along $\bar{L}$, is not $\perp$ to z-axis!)
[For later discussion, we point out that these
2 components are $\dot{\phi}$, i.e., rotation about
space z-axis ($\tilde{\bar{e}}_3$), and $\dot{\psi}$, i.e., rotation about $\bar{e}_3$:
$\bar{\omega} = \dot{\phi}\, \tilde{\bar{e}}_3 + \dot{\psi}\, \bar{e}_3 + \dot{\theta}\, \bar{e}_1'$ line of nodes
vanishes here ]

— The latter/above "component" of $\bar{\omega}$ along z-axis does not give any displacement of z-axis of top ⇒ former component along $\bar{L}$ must be the $\omega_{pr}$ needed

— Now, from figure, we see that

$\omega_{pr} \times \sin\theta = \omega_2$ since rest of $\bar{\omega}$ (along z-axis has no component along <u>y</u>-axis). But,

$\omega_2 = L_2/I_1 = L\sin\theta/I_1 \Rightarrow \boxed{\omega_{pr.}} = \frac{L}{I_1}$ ($=$ constant) ... (3)

— We can be $\boxed{\text{ambitious}}$ by deducing additional results (as follows)! Clearly $|\bar{\omega}|^2 = \omega_3^2 + \omega_1^2$

$= L^2\left(\cos^2\theta/I_3^2 + \sin^2\theta/I_1^2\right) =$ constant.

— Combining this with $\omega_3$ being constant shows that $\bar{\omega}$ is at fixed <u>angle</u> relative to $\boxed{z}$-axis ($\theta''$ in figure). Since angle between z-axis and $\bar{L}$ was already shown to be constant, so is that <u>between</u> $\boxed{\bar{\omega}}$ and $\boxed{\bar{L}}$ ($\theta'$ in figure), i.e., $\theta' + \theta'' = \theta$ (it's crucial to use here $\bar{\omega}$, $\bar{L}$ and z-axis being in same plane)

— We can <u>relate</u> $\boxed{\theta'}$ to $\boxed{\theta''}$ (thus obtaining both in terms of $\theta$) as follows. <u>Return</u> to decomposition of $\bar{\omega}$ into $\omega_{pr}$ (along $\bar{L}$) [$= L/I_1$] and along top axis; <u>latter is given</u> by $\left[\omega_3 - L\cos\theta/I_1\right]$, i.e., we should get spin of top ($\omega_3$) when the latter component is $\boxed{\text{added}}$ to component of $\omega_{pr}$ ($= L/I_1$) along z-axis [again, for later discussion, this latter component is $\dot{\psi} = \omega_3 - \dot{\phi}\cos\theta$]

— Now, we must have

$$\underbrace{L/I_1}_{\substack{\omega_{pr} \\ \bar{\omega} \text{ along } \bar{L}}} \times \underbrace{\sin\theta'}_{\substack{\text{to give} \\ \text{component } \perp \\ \text{to } \bar{\omega}}} = \underbrace{\left(\omega_3 - \cos\theta \frac{L}{I_1}\right)}_{\substack{\text{"rest" of } \bar{\omega} \\ \text{(along z-axis)}}} \times \underbrace{\sin\theta''}_{\substack{\text{to give} \\ \text{component} \\ \perp \text{ to } \bar{\omega}}}$$

i.e, net component of $\bar{\omega}$ $\perp$ to $\bar{\omega}$ $= 0\ (!) \Rightarrow$

$$\boxed{\sin\theta'/\sin\theta''} = \frac{\left(\omega_3 - \frac{L}{I_1}\cos\theta\right)}{L/I_1} = \frac{L\cos\theta\left(\frac{1}{I_3} - \frac{1}{I_1}\right)}{L/I_1}$$

use Eq.(2)

$$= \cos\theta\left(I_1/I_3 - 1\right) \quad \ldots (4)$$

—— x ——

$\boxed{\text{(II)}}$. Next, connect (or match) to $\boxed{\text{Euler angles}}$

— choose $\bar{L}$ to be along <u>space</u> $\bar{z}$-axis $(\bar{\tilde{e}}_3)$ so that polar angle of <u>top axis</u> $(\bar{e}_3)$ w.r.t. $\bar{\tilde{e}}_3$ is $\theta$,

— whereas its <u>azimuthal angle</u> is $(\phi - \pi/2)$: this can be "seen" from figure below, i.e. by dropping perpendicular from tip of $\bar{e}_3$ axis onto $\bar{\tilde{e}}_1$ - $\bar{\tilde{e}}_2$ (i.e, space x-y) plane or explicitly, use $\bar{\tilde{e}} = R^{-1}\bar{e} = R^T\bar{e}$ with $\mathbf{e} = (0\ 0\ 1)^T$, i.e, find $\bar{e}_3$ components along space axes:

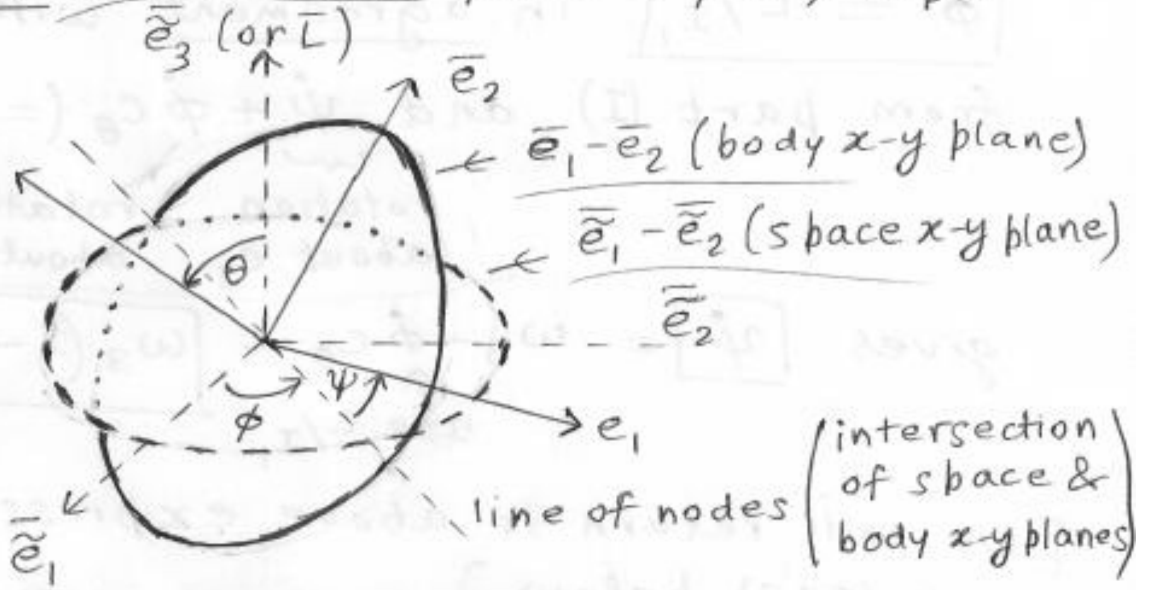
$$\bar{e}_3 = s_\theta s_\phi \bar{\tilde{e}}_1 - s_\theta c_\phi \bar{\tilde{e}}_2 + c_\theta \bar{\tilde{e}}_3$$

$$= s_\theta \underbrace{\cos(\phi - \pi/2) \bar{\tilde{e}}_1 + s_\theta \sin(\phi - \pi/2)}_{\text{cos/sin of azimuth}} \bar{\tilde{e}}_2 + c_\theta \bar{\tilde{e}}_3$$

— Also, note that $\boxed{\text{line of nodes}}$ (called earlier $\bar{e}_1'$) $\perp$ <u>both</u> $\bar{e}_3$ & $\bar{\tilde{e}}_3$

— Thus, $\boxed{\dot{\phi}}$ corresponds to angular velocity of precession discused in part I, i.e., $\omega_{pr}$

(similar to Fig. 47 on page 110 of LL)




— Onto formulae: we have (see HW 6.5) again

$\omega_1$ ($\bar{\omega}$ component along body x-axis) $= \dot{\phi} s_\theta s_\psi + \dot{\theta} c_\psi$;

$\omega_2 = \dot{\phi} s_\theta c_\psi - \dot{\theta} s_\psi$ and $\omega_3 = \dot{\phi} c_\theta + \dot{\psi}$

— As before, we can use symmetry about z-axis of top ($\bar{e}_3$) in order to choose at $\boxed{\text{given}}$ instant (see later for what happens at "next" instant!) x-y axes such that $\boxed{\psi = 0}$ or $\boxed{\bar{e}_1}$ is line of $\boxed{\text{nodes}}$ so that

$\omega_1 = \dot{\theta}$ ; $\omega_2 = \dot{\phi} s_\theta$ (while $\omega_3 = \dot{\psi} + \dot{\phi} c_\theta$) ... (5)

— On the other hand, we have (with $\psi = 0$)

$L_1 = 0$ (since $\bar{e}_1$ chosen to be $\perp$ to $\tilde{\bar{e}}_3$, i.e., $\bar{L}$);

$L_2 = L \sin\theta$ and $L_3 = L\cos\theta$ ... (6)

— Matching (5) & (6) [using $L_a = I_a \omega_a$ (no sum over $a$!)]

we get $\dot{\theta} \; (= \omega_1) = L_1 / I_1 = 0$, i.e., $\boxed{\theta = \text{constant}}$;
[as in Eq. (1) of part (I)]

$\dot{\phi}\, s_\theta \,(= \omega_2) = L_2/I_1 = L \sin\theta / I_1 \Rightarrow$ (6)

$\boxed{\dot{\phi} = L/I_1}$ in <u>agreement</u> with $\omega_{pr}$ in Eq.(3)

from part (I) and $\underbrace{\dot{\psi}}_{\substack{\text{rotation}\\ \text{about } \bar{e}_3}} + \underbrace{\dot{\phi}\, c_\theta}_{\substack{\text{rotation}\\ \text{about } \bar{\bar{e}}_3}} (= \omega_3) = L_3/I_3 = \dfrac{L c_\theta}{I_3}$

gives $\boxed{\dot{\psi}} = \omega_3 - \underset{\substack{\uparrow \\ \text{use } L/I_1}}{\dot{\phi}}\, c_\theta = \boxed{\omega_3 \left(1 - I_3/I_1\right)} \ldots (7)$

[we will return to above expression <u>for $\dot{\psi}$ in</u> part (III) below.]

— It is also clear that $\boxed{\bar{\omega}, \bar{L} \text{ \& top axis}}$ are all in <u>one plane</u> ($\perp$ to $\bar{e}_1$ / line of nodes)

[Of course, we also (re-)obtain $|\bar{\omega}| = \sqrt{\omega_3^2 + \omega_2^2}$ = constant etc. in this way.] (again, body axes components)

— Now, at $\boxed{\text{next}}$ <u>instant</u>, say after time $\delta t$, we of course get a non-zero (even if small) $\psi$, i.e., $\boxed{\delta\psi = \dot{\psi}\,\delta t}$ ($\psi > 0$ is "anticlockwise")

— <u>However</u>, using the same "freedom" in choosing body (principal) x-y axes, we can (again) "re-define" (slightly) body x-y axis so as to get back earlier picture: concretely, we choose <u>new</u> x-axis (call it $\bar{e}_1^{(new)}$) to be rotated w.r.t. old x-axis $\bar{e}_1$ by $(-\delta\psi)$, i.e., $\delta\psi$ in clockwise direction.

[this is just rotation of body x-y plane <u>about</u> $\bar{e}_3$]

In this way, $\bar{e}_1^{(new)}$ is still along line of nodes so that $\psi^{(new)} = 0 \Rightarrow L_1^{new} = 0$ etc. (as before), i.e., $\boxed{\omega_1^{(new)} = 0}$

— In other words, $\boxed{\text{if}}$ we keep "changing" (7)
body x-y axes (as body itself is moving in space frame) in above fashion [i.e, with "angular velocity" of $\Theta\dot{\psi}$ about top axis ($\bar{e}_3$)] then $\bar{\omega}$ seems to be "fixed" in body frame [i.e., $\omega_3 = |\bar{\omega}| \cos\theta$ ; $\omega_2^{(new)} = |\bar{\omega}| \sin\theta$ & $\omega_1^{(new)} = 0$]!

— However, above rotation / re-definition of body x-y axes was merely a "trick" (e.g., designed to show $\dot{\theta} = 0$, compute precession etc.): in reality, we work with fixed body x-y axes (like "new" body axes) $\Rightarrow$ $\boxed{\bar{\omega}}$ is actually rotating with angular velocity $\boxed{-\dot{\psi}}$ about axis of top (again, this is body-frame viewpoint): we will return to this point just below.

—— × ——

— Again, let's do more checks (as follows): resolve $\bar{\omega}$ along space-frame axes (cf. body frame done earlier): from HW 6.5, we get [for our case of $\dot{\theta} = 0$] $\tilde{\omega}_1$ (again, $\bar{\omega}$ component along space x-axis) $= \dot{\psi} s_\theta s_\phi$ ; $\tilde{\omega}_2 = -\dot{\psi} s_\theta c_\phi$ and $\tilde{\omega}_3 = \dot{\psi} c_\theta + \dot{\phi}$.

— So, we find (i) $|\bar{\omega}| = \sqrt{\tilde{\omega}_1^2 + \tilde{\omega}_2^2 + \tilde{\omega}_3^2} = \sqrt{\dot{\phi}^2 + \dot{\psi}^2 + 2c_\theta \dot{\psi}\dot{\phi}}$, i.e, constant and (ii) azimuthal angle of $\bar{\omega}$ is $(\phi - \pi/2)$ [since $\tilde{\omega}_2 / \tilde{\omega}_1 = \tan(\phi - \pi/2)$], i.e, $\boxed{\bar{\omega}}$ rotates about space z axis at rate of $\dot{\phi}$

**(III)**. Finally, we use <u>Euler's</u> $\boxed{\text{equations}}$, i.e., for time-dependence of components of $\bar{\omega}$ along body axes. We have (with $\boxed{I_1 = I_2}$)

$I_3 \dot{\omega}_3 = 0 \Rightarrow \boxed{\omega_3 = \text{constant}}$ (as before)
(i.e., spin/rotation about top axis)

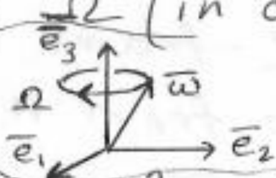
$\Rightarrow \boxed{\cos\theta} = L_3 \left(= I_3 \omega_3\right) / |\mathbf{L}| \boxed{= \text{constant}}$

constant (again, as before)

[Note that $I_1 = I_2$ is crucial here: for $I_1 \neq I_2$ (asymmetric top), we have (in general) $I \dot{\omega}_3 = -\omega_1 \omega_2 (I_2 - I_1) \neq 0$, i.e., $\omega_3 \neq$ constant]

— However, $\dot{\omega}_{1,2} \neq 0$, i.e., $I_1 \dot{\omega}_1 = + \omega_2 \Omega I_1$,
where $\boxed{\Omega}$ (as in <u>DT's</u> notes) $\equiv \omega_3 (I_1 - I_3)/I_1$
note GPS, LL have opposite sign here
constant ...(8)
and $\dot{\omega}_2 = -\omega_1 \Omega \Rightarrow \ddot{\omega}_1 = -\Omega^2 \omega_1$

— Solution is $\omega_{1,2} = \overbrace{\omega_0}^{\text{constant}} \left(\sin \Omega t, \cos \Omega t\right)$,
i.e., <u>component</u> of $\boxed{\bar{\omega}}$ $\perp$ to <u>axis</u> of top $[= \omega_1 \bar{e}_1 + \omega_2 \bar{e}_2]$ rotates about top axis (itself rotating in space frame) with angular velocity $\Omega$ (in clockwise direction for $\Omega > 0$):




(Fig. 30 on page 56 of DT)

$\Rightarrow \boxed{|\bar{\omega}|^2} = \omega_3^2 + \omega_1^2 + \omega_2^2 = \omega_3^2 + \omega_0^2 \boxed{= \text{constant}}$

— Since $\omega_3$ ( = component <u>along</u> $\bar{e}_3$) is also constant, it is clear that <u>entire</u> $\bar{\omega}$ rotates about axis of top at rate $\Omega$

— Since $L_{1,2} = \underbrace{I_{1,2}}_{\substack{\text{same} \\ \text{(constant)}}} \underbrace{\omega_{1,2}}_{\substack{\text{different} \\ \text{(changing} \\ \text{with time)}}}$ and $L_3 = I_3 \omega_3$ = constant, we also see that (entire) $\bar{L}$ rotates about axis of top with angular velocity $\Omega$ (9)

[ again, this is motion of $\bar{L}$ – as "defined/constructed" by space/inertial frame observer, i.e., $\sum_i m_i \bar{r}_i \times \frac{d\bar{r}_i}{dt}$ as observed in space frame – but as resolved along body axes (or "seen" by body observer, i.e., $\bar{L} = \sum_a L_a (\bar{e}_a)$, constant for body observer; $L_3 =$ constant, but $\dot{L}_{1,2} = I_{1,2}\dot{\omega}_{1,2} \neq 0$ ]

— x —

— Match the above rotation of $\boxed{\bar{\omega}}$ (or $\bar{L}$) about top axis as obtained from Euler's equations in part (III) just above to what we earlier concluded in part (II) using Euler angles, i.e., $\boxed{\Omega}$ "should" equal $\boxed{\dot{\psi}}$ (both clockwise): indeed Eq.(7) for $\dot{\psi}$ is same as Eq.(8) for $\Omega$

— Equivalently (more explicitly), in body frame we can look at rotation of $\boxed{\bar{L}}$, which is along $\tilde{\bar{e}}_3$. So, expand $\tilde{\bar{e}}_3$ in $\bar{e}_{1,2,3}$ [using $\bar{e} = R\tilde{\bar{e}}$ and setting $\tilde{\bar{e}} = (0,0,1)^T$ ] giving

$$\tilde{\bar{e}}_3 = S_\theta S_\psi \bar{e}_1 + S_\theta C_\psi \bar{e}_2 + C_\theta \bar{e}_3$$

(constant)

shows clockwise rotation about $\bar{e}_3$ at rate $\dot{\psi}$ about $\bar{e}_3$

i.e., $\bar{L}$ rotates (clockwise) about top axis at rate $\dot{\psi}$, (again) matching $\Omega$ obtained in part (III).

— Finally, we can come a "full circle" by seeing the above rotation of $\bar{\omega}$ (or $\bar{L}$) about top axis (i.e., from body frame/viewpoint)

— It's $\boxed{\text{easier}}$ to consider $\boxed{\bar{L}}$, which is fixed in space-frame (cf. $\boxed{\bar{\omega}}$ moving in space-frame also!).

— Begin with $\omega_{prec}$ component of $\bar{\omega}$ (i.e., along $\bar{L}$ / vertical space axis): it gives an instantaneous velocity to entire body $x$-axis (coming out of $z$-$\bar{L}$ plane) along horizontal/right in figure on page 1.
$\Rightarrow$ body $x$-axis remains $\perp$ to $\bar{L}$ in this process, i.e., $L_1$ stays 0 (also, $L_2 = L \sin\theta$ and $L_3 = L\cos\theta$ even for new $y$, $z$ axis): clearly $\bar{L}$ does $\boxed{\text{not}}$ move even in body-frame (again, on account of $\omega_{prec}$ only)

— There remains "rest" of $\bar{\omega}$ component (see figure on page 1) along top-axis: as discussed before, $\boxed{\text{this}}$ does not give any displacement of top (since this $\bar{\omega} \parallel \hat{z}$-axis). It corresponds to simply rotation of body $x$-$y$ plane (in space frame of course!) about $z$-axis, while $\bar{L}$ is fixed

— Thus, for an observer $\boxed{\text{fixed}}$ in body, it is $\bar{L}$ which is rotating at rate of $\ominus$ "rest" of $\bar{\omega}$.

— Now, "rest" of $\bar{\omega}$ (see bottom of page 3/top of page 4)
$= \omega_3 - L\cos\theta/I_1 = \omega_3 - L_3/I_1 = \omega_3 - I_3\omega_3/I_1 = \omega_3\left(1 - \frac{I_3}{I_1}\right)$
so that $\bar{L}$ is rotating in body frame about $z$-axis "clockwise" at rate of $\omega_3(1 - I_3/I_1)$, matching $\dot{\psi}$ of (II) and $\Omega$ of (III)