

Phys 601 by Dr. Agashe

mostly (1)

Rigid Body Motion

Based on chapter 3 of DT and chapters 4, 5 of GPS

— So far, we studied motion of isolated/independent point particles

— Onto extended objects/rigid bodies, i.e.,

collection of N points so that distance between any pair is fixed: $|\vec{r}_i - \vec{r}_j| = \text{constant} (\forall i, j)$,

although direction of $\vec{r}_i - \vec{r}_j$ can change

— we can transition from discrete/set of particles to continuous system simply by

$$\sum_i m_i \rightarrow \int d\vec{r} \rho(\vec{r})$$

density of rigid body

— A rigid body has 6 degrees of freedom (d.o.f.), 3 of which describe translation, while 3 are rotational

— To start with, we will focus on rotational motion only

— Outline of formalism part

— kinematics: \vec{v} (linear velocity) of point particle

this note \rightarrow $\vec{\omega}$ (angular velocity) of rigid body

Plan: simple/intuitive picture, followed by formal, more mathematical language

→ next note

linear momentum (2)

- dynamics: for point particle, $\vec{F} = d\vec{p}/dt$,
 where $\vec{p} = m\vec{v}$ → for rigid body λ ^{rotation}

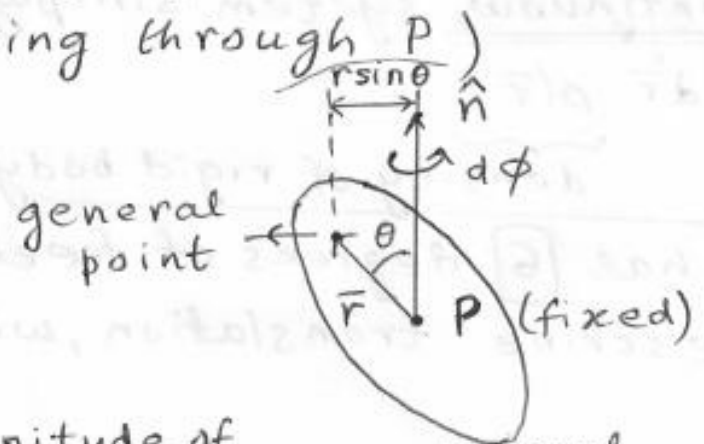
$\vec{\tau}$ (torque) = $\frac{d\vec{L}}{dt}$, where $L = I\vec{\omega}$

↑ angular momentum

↑ moment of inertia
(will be generalized to tensor)

— Angular velocity (section 3.1 of DT) (warm-up)

— Usual / simple language: rigid body fixed at point P (can be taken to be center-of-mass), rotating by (small) angle $d\phi$ about axis \hat{n} (passing through P)



magnitude of displacement of general point with position vector \vec{r} is given by $|d\vec{r}| = |\vec{r}| \sin \theta d\phi$

(small) part of radius of circle along which point moves

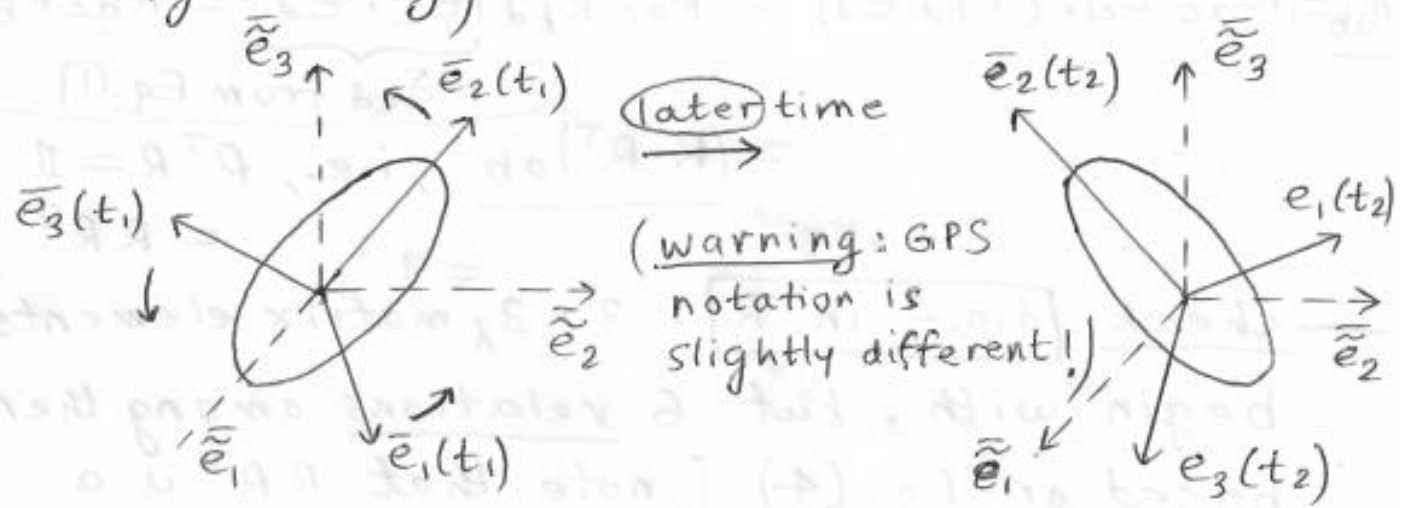
and direction of $d\vec{r}$ is \perp to \vec{r} and \hat{n}
 (again, distance of general point to P is fixed)

$$\Rightarrow d\vec{r} = \underbrace{d\phi}_{\hat{n} d\phi} \times \vec{r} \quad \text{or} \quad \dot{\vec{r}} = \underbrace{\vec{\omega}}_{\substack{\uparrow \text{(usual)} \\ d\phi/dt}} \times \vec{r} \dots (1)$$

Tale of 2 frames

Onto more formal/mathematical notation:

consider two frames/coordinate systems shown below, i.e., fixed/external to body/space frame denoted by $\tilde{e}_{a=1,2,3}$ and moving/body frame \bar{e}_a (latter rotates with the rigid body)



[For concreteness, \bar{e}_a or \tilde{e}_a 's could just be unit vectors along x, y, z axes.] as shown above

We can always choose both axes to be orthogonal:

$$\bar{e}_a(t) \cdot \bar{e}_b(t) = \delta_{ab} = \tilde{e}_a \cdot \tilde{e}_b \dots (2)$$

↑
←
←
←

moving with body time-independent (convenient)

[Anticipating (later, there is a natural choice for body axes such that inertia tensor is diagonal.)

We can express body frame unit vectors ψ in terms of space frame via 3x3 matrix, R_{ab} :

$$\bar{e}_a(t) = \sum_b R_{ab}(t) \bar{e}_b \quad \text{--- constant} \quad (4)$$

— clearly, R is orthogonal, i.e.,

$$R^T R = \mathbb{1} \quad \dots (4)$$

[Explicitly, we have $\bar{e}_a \cdot \bar{e}_b = \delta_{ab}$ from Eq.(1)]

so that plugging in Eq.(3) gives

$$\delta_{ab} = (R_{ac} \bar{e}_c) \cdot (R_{bd} \bar{e}_d) = R_{ac} R_{bd} (\bar{e}_c \cdot \bar{e}_d) = R_{ac} R_{bd} \delta_{cd}$$

$$= (R R^T)_{ab} \quad \text{--- } \delta_{cd} \text{ from Eq.(1)}$$

$$= \mathbb{1} \quad \text{--- } R^T R = \mathbb{1}$$

$$= R R^T \quad \text{--- } R^T R = \mathbb{1}$$

— check d.o.f. in R : 3×3 matrix elements to

begin with, but 6 relations among them based on Eq.(4) [note that $R R^T$ is a symmetric matrix so that it has 6 independent elements] $\Rightarrow 9 - 6 = 3$ d.o.f. in R , as expected since R parametrizes rotation of body (frame)

— In other words, we can "replace" moving/body frame by 3×3 orthogonal matrix, $R(t)$

\Rightarrow we expect angular velocity ($\bar{\omega}$) to be related to $\dot{R} (\equiv dR/dt)$

— Let's work out $\bar{\omega} - R, \dot{R}$ relation; what follows is lot of algebra (so, brace yourselves!); end results are in Eqs.(7),(8)

we can expand/express position vector of any point in rigid body either in terms of unit vectors of space/fixed or moving/body frame:

we'll do analogously for other vectors later

$$\vec{r}(t) = \begin{cases} \sum_a \tilde{r}_a(t) \vec{e}_a & \text{in space/fixed frame} \\ \sum_a r_a \vec{e}_a(t) & \text{in moving/body frame} \end{cases}$$

independent of time ... (5)(a)

Again, coordinates of point relative to body axes are fixed (being a rigid body), but these axes are themselves moving in space frame.

[If you'd like to be concrete, take

$$\vec{r} = \tilde{x} \vec{i} + \tilde{y} \vec{j} + \tilde{z} \vec{k} = x \vec{i} + y \vec{j} + z \vec{k}$$

all ^{are} time dependent

where $\vec{i}, \vec{j}, \vec{k}$ are unit vectors along Cartesian axes in body frame (similarly $\vec{i} \dots$ in spaceframe)

note that $\tilde{r}_b(t) = r_a R_{ab}(t)$ as per Eq.(3)

and above in spaceframe inertial

So, Eq.(5) gives $\frac{d\vec{r}}{dt} = \begin{cases} \sum_a (d\tilde{r}_a/dt) \vec{e}_a & \text{in spaceframe} \\ \sum_a r_a d\vec{e}_a/dt & \text{in body frame} \end{cases}$

note this is velocity $\leftarrow \frac{d}{dt}$ seen by inertial observer, but can be resolved in either set of axes ... (5)(b) $\sum_b \frac{dR_{ab}}{dt} \vec{e}_b$ as per Eq.(3)

consider how body frame itself changes with time (i.e., if you wish, set $r_a = 1, r_{b,c} = 0$ above)

$$d\vec{e}_a/dt = \sum_b dR_{ab}/dt \vec{e}_b = \sum_{b,c} \frac{dR_{ab}}{dt} (R^{-1})_{bc} \vec{e}_c$$

i.e., expressed in body frame axes. use inverse of Eq.(3)

$$\equiv \sum_c \Omega_{ac} \bar{e}_c \quad \dots \text{[6]} \quad \text{[6]}$$

$$\boxed{\Omega_{ac}} \equiv \sum_b \dot{R}_{ab} (R^{-1})_{bc} = \sum_b \dot{R}_{ab} R_{cb} \dots \text{[7]} \quad \leftarrow \text{use } R^{-1} = R^T \text{ from Eq. (4)}$$

— It looks like we have a tensor thus far, but we can "reduce" it to a vector (i.e., 3 components) by realizing that Ω is antisymmetric:

$$\sum_b R_{ab} \underbrace{R_{cb}}_{(R^T)_{bc}} = \delta_{ac} \xrightarrow{\text{taking derivatives}} \sum_b \underbrace{\dot{R}_{ab} R_{cb}}_{\Omega_{ac}} + \underbrace{R_{ab} \dot{R}_{cb}}_{\Omega_{ca}} = 0$$

↑ use Eq. (4) ← use Eq. (7)

— So, we define a vector, $\bar{\omega}$ (formal) expressed in body frame as

$$\bar{\omega}_{(\text{formal})} = \sum_a \omega_a \bar{e}_a, \quad \text{where } \omega_a = \sum_{b,c} \frac{1}{2} \epsilon_{abc} \Omega_{bc} \dots \text{[8]}$$

[Recall 3x3 antisymmetric matrix has only 3 independent elements.] $\bar{\omega}$ is same as usual one in order to show (formal)

— Then, it is "straightforward" to re-write Eq. (6) as

$$d\bar{e}_a/dt = \sum_{b,c} \epsilon_{abc} \omega_b \bar{e}_c = \bar{\omega} \times (\bar{e}_a) \quad [\text{not } (\bar{\omega} \times \bar{e})_a!]$$

\dots \text{[9]}

[check (can skip in lecture): plug $\omega_b = \frac{1}{2} \epsilon_{bde} \Omega_{de}$ from Eq. (8) into middle of Eq. (9) above to get

$$\begin{aligned} -\epsilon_{abc} \omega_b &= -\frac{1}{2} \epsilon_{abc} \epsilon_{bde} \Omega_{de} = -\frac{1}{2} (\epsilon_{bca} \epsilon_{bde}) \Omega_{de} \\ &= -\frac{1}{2} (\delta_{cd} \delta_{ae} - \delta_{ce} \delta_{ad}) = -\frac{1}{2} (\Omega_{ca} - \Omega_{ac}) \\ &= +\Omega_{ac} \end{aligned}$$

where standard properties of ϵ have been used.]

— Plug Eq. (9) into 2nd line of Eq. (5) (b) to get

$$\dot{\vec{r}} = \sum_a r_a \dot{\vec{e}}_a(t) = \sum_a r_a \left[\bar{\omega} \times (\vec{e}_a) \right] = \bar{\omega} \times \underbrace{\sum_a r_a \vec{e}_a}_{\vec{r}} \quad (7)$$

↑
constant

i.e., same as Eq. (1) obtained using simple/usual picture. Thus, the 2 definitions of $\bar{\omega}$ [simple used in Eq. (1) and more formal one in terms of Eq. (7), (8)] are indeed identical.

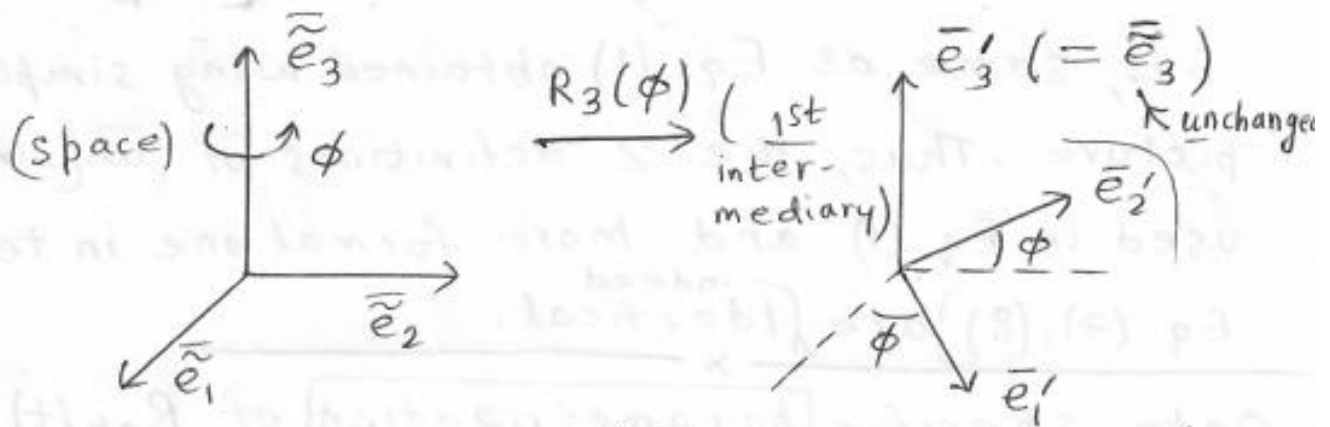
— Onto specific parametrization of $R_{ab}(t)$:

Euler angles (sec. 3.5 of DT)

- so far, a bit abstract (!!); let's give^a face to 3×3 orthogonal matrix relating 2 frames
- Euler's theorem: arbitrary rotation can be expressed as product of (3) successive rotations (3 angles) about 3 (in general different) axes
- Again, we want to go from \vec{e}_a (space frame) to $\bar{\vec{e}}_a$ (body frame) via 3×3 orthogonal matrix (R) parametrized by 3 (Euler) angles (as follows, which is lot of algebra; so beware!)
- 3 steps/^{"component"} rotations so that we'll have 2 intermediary (again!) frames: schematically

\vec{e}_a	$\xrightarrow{R_3(\phi)}$	\vec{e}'_a	$\xrightarrow{R_1(\theta)}$	\vec{e}''_a	$\xrightarrow{R_3(\psi)}$	$\bar{\vec{e}}_a$
(space)		(1 st intermediary)	(1 st axis by θ)	(different than 1 st step)	(3 rd axis by ψ)	(body)
	rotate about 3 rd axis by ϕ					

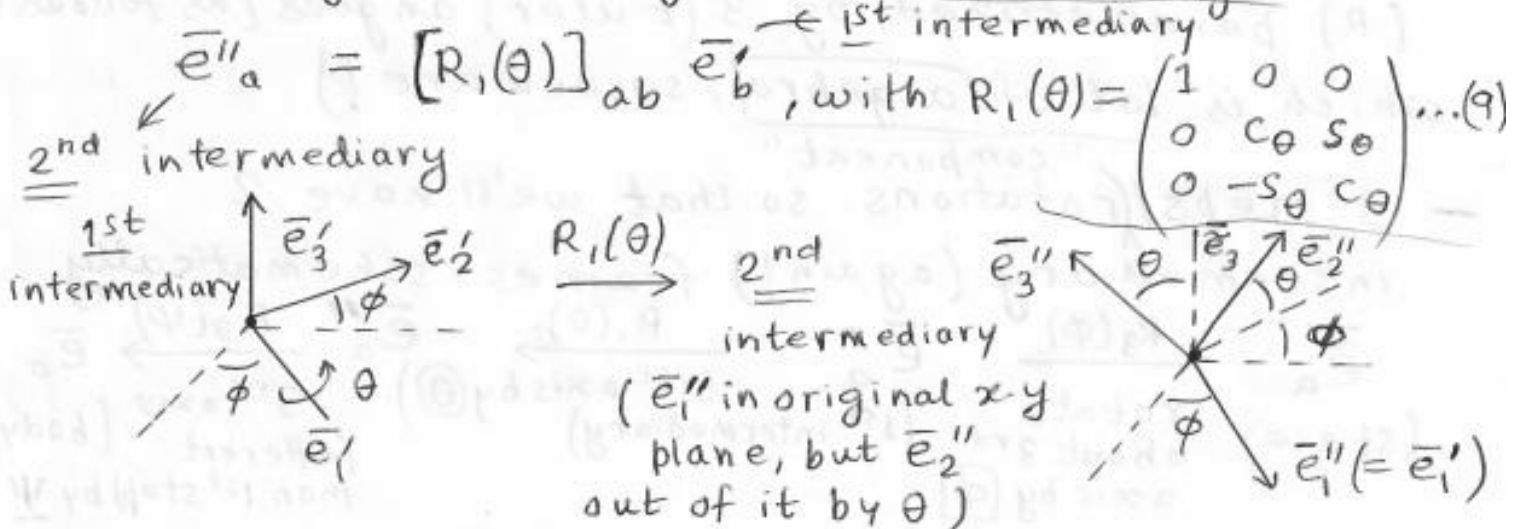
- In more/gory detail, we have in ϵ^{12} (8)
1st step (rotate by ϕ about \bar{e}_3 axis):



- clearly z component of a vector is unchanged, (\bar{e}_2 still in original x - y plane)
 whereas x, y just get rotated by ϕ into each other so that ... (8)

$\bar{e}'_a = [R_3(\phi)]_{ab} \bar{e}_b$, where $R_3(\phi) = \begin{pmatrix} c\phi & s\phi & 0 \\ -s\phi & c\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$
 (where $c\phi \equiv \cos\phi$ and $s\phi \equiv \sin\phi$)

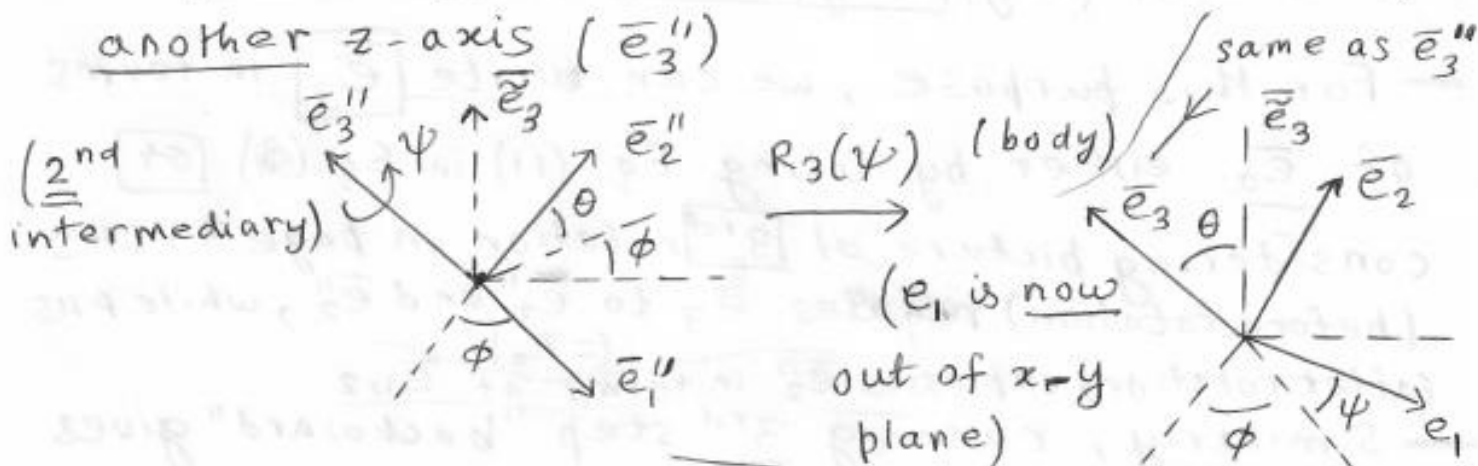
- 2nd step [rotate by θ about (new) x -axis, i.e., \bar{e}'_1 (called line of nodes) so that z -component now changes (so does y), but x is unchanged]:



- 3rd step [rotation by ψ about (new) \underline{z} -axis, (9)
 i.e., \bar{e}_3''] so that \bar{e}_3'' is 2nd intermediary

$$\bar{e}_a \text{ (body)} = [R_3(\psi)]_{ab} \bar{e}_b'', \text{ with } R_3(\psi) = \begin{pmatrix} c_\psi & s_\psi & 0 \\ -s_\psi & c_\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \dots (10)$$

i.e., "like" 1st step, but about another z -axis (\bar{e}_3'')



- So, putting it all together (in order):

$$R_{ab}(\phi, \theta, \psi) = \left[\underset{\substack{\uparrow \\ \text{3rd rotation}}}{R_3(\psi)} R_1(\theta) \underset{\substack{\uparrow \\ \text{1st rotation}}}{R_3(\phi)} \right]_{ab} \left[\text{use Eqs. (8)-(10)} \right]$$

$$= \begin{pmatrix} c_\psi c_\phi - c_\theta s_\phi s_\psi & s_\theta c_\psi + c_\theta s_\psi c_\phi & s_\theta s_\psi \\ -c_\phi s_\psi - c_\theta c_\psi s_\phi & -s_\psi s_\phi + c_\theta c_\psi c_\phi & s_\theta c_\psi \\ s_\theta s_\phi & -s_\theta c_\phi & c_\theta \end{pmatrix} \dots (11)$$

- Finally, let's express $\underline{\bar{\omega}}$ in terms of Euler angles' time derivatives

- Brute force way: plug Eq. (11) into Eqs. (7), (8)

- Quicker: ^{considering} (small) changes $d\phi$, $d\theta$ & $d\psi$ in dt , we get

$$\bar{\omega} = \dot{\phi} \bar{e}_3 + \dot{\theta} \bar{e}'_1 + \dot{\psi} \bar{e}_3 \dots (12) \quad (10)$$

\downarrow since ϕ was rotation about \bar{e}_3 (space)
 \downarrow θ about e'_1 (x-axis of intermediaries)
 \downarrow ψ is about e_3 (body)

— However, we would like to express $\bar{\omega}$ in terms of (only) body frame unit vectors

— For this purpose, we can write \bar{e}_3 in terms of \bar{e}_a either by using Eq. (11) in Eq. (3) or considering picture of 3rd rotation on page 9: LHS (before rotation) relates \bar{e}_3 to \bar{e}_3'' and \bar{e}_2'' , while RHS (after rotation) expresses \bar{e}_2'' in terms of $\bar{e}_{1,2}$ ($= \bar{e}_3$)

— Similarly, running 3rd step "backward" gives $e'_1 (= \bar{e}_1'')$ in terms of $\bar{e}_{1,2}$ (end of 3rd step)

\downarrow end of 1st step \downarrow end of 2nd step

— Rest is left for HW 6.3 (a)

— Similarly, we can express $\bar{\omega}$ in terms of \bar{e}_a (i.e., space-frame axes): see HW 6.3 (b)