$\boxed{\text{Phys 601}}$ Dr. Agashe

Going to $\boxed{\text{principal axes}}$ of rigid body

— Suppose we choose arbitrarily the body set of axes to begin with (with fixed point in body, P, being origin)

— Then, in general, intertia tensor ($\overleftrightarrow{I}$) with elements:

$$I_{ab} = \underbrace{\sum_i}_{\text{points/particles in body}} m_i \left[|\bar{r}_i|^2 \delta_{ab} - (r_i)_a (r_i)_b \right] \quad \ldots (1)$$

will be off-diagonal

— Can we find another set of body axes (origin still at P) such that inertia tensor is diagonal?

— First, let us determine how $\overleftrightarrow{I}$ (to begin with) transforms when we perform an arbitrary rotation of body axes (again, both axes are moving with the body), i.e., in terms of unit body-frame vectors, we have (schematically)

$$\bar{e}_{new} = O \cdot \bar{e}_{old}, \text{ where } \underbrace{O^T}_{\text{same as } O^{-1}} O = \mathbb{1} \quad \ldots (2)$$

i.e., O is 3×3 orthogonal matrix

— For position vector of a point in the body, we get

$$\bar{r} = \sum r_a^{new} \bar{e}_{new}^{\,a} = \sum_a r_a^{old} \bar{e}_{old}^{\,a} \ldots (3)$$

— Using inverse of Eq. (2), i.e., $\bar{e}_{old} = O^T \bar{e}_{new}$ or explicity $(\bar{e}_{old})_a = (O^T)_{ab} (\bar{e}_{new})_b = O_{ba} (\bar{e}_{new})_b$, in Eq. (3), we get

$$\sum_a r_a^{new} \bar{e}_{new\,a} = \sum_a r_a^{old} O_{ba} (\bar{e}_{new})_b$$

$\leftarrow$ a, b are dummy indices

$$= \sum_b r_b^{old} O_{ab} (\bar{e}_{new})_a$$

— Matching coefficient of $(\bar{e}_{new})_a$ on 2 sides gives

$$r_a^{new} = \sum_b r_b^{old} O_{ab} \quad \ldots (4)$$

— Now, inertia tensor with respect to $\boxed{\text{new}}$ body axes is given by Eq. (1), but with $\mathbf{r}_{\underline{new}}$, i.e.,

$$(I_{\underline{new}})_{ab} = \sum_i m_i \left[|\bar{r}_{new\, i}|^2 \delta_{ab} - (r^i_{new})_a (r_i^{new})_b \right] \quad \ldots (5)$$

— Clearly $|\bar{r}_{i\, new}|^2 = |\bar{r}_{i\, old}|^2$ and $\delta_{ab} = (O\,O^T)_{ab}$

$$= O_{ac} (O^T)_{cb}$$

$$= O_{ac} \delta_{cd} (O^T)_{db}.$$

— Using above 2 relations in 1st term of Eq. (5) and Eq. (4) twice in 2nd term, we get

$$(I_{new})_{ab} = \sum_i m_i \left[|\bar{r}_{old\, i}|^2 O_{ac} \delta_{cd} (O^T)_{db} - \underbrace{(r_c^{old})_i O_{ac}}_{r_a^{new}} \times \underbrace{(r_d^{old})_i O_{bd}}_{r_b^{new}} \right]$$

$$= O_{ac} (I^{old})_{cd} (O^T)_{db}, \text{ i.e., } \overleftrightarrow{I}_{new} = O\, \overleftrightarrow{I}_{old}\, O^T \quad \ldots (6)$$

— Finally, since $\overleftrightarrow{I}(old)$ is symmetric, it can be diagonalized by orthogonal transformation, i.e, there exists O_{I}:

$$\underbrace{O_I \overleftrightarrow{I}_{old} O_I^T}_{\text{(arbitrary thus far)}} = \underbrace{\text{diag}(I_1, I_2, I_3)}_{\text{(specific)}} \ldots (7) \quad [O_I O_I^T = \mathbb{1}]$$

— Setting O in Eq. (6) equal to O_I of Eq. (7) gives

$$\overleftrightarrow{I}_{new} = \text{diag}(I_1, I_2, I_3) \text{ as desired}$$