Phys 601 Dr. Agashe

Heavy, symmetric top ($I_1 = I_2 \neq I_3$): Specific cases (based on GPS section 5.7 and DT section 3.6)

(1). Uniform precession ($\dot{\phi}$ = constant), without nutation ($\dot{\theta} = 0$), like free top

— We would like to see whether $\theta = \theta_0$ = constant ($\dot{\theta} = 0$) is allowed (inspite of gravity tending to make top fall)

— First of all, plugging above into expression for either p_ϕ or E (being constants) gives $\dot{\phi}$ = constant

— Now, $\dot{u} = -S_\theta \dot{\theta}$ and $\dot{u}^2 = f(u)$ [see previous note] implies that $u_0 \equiv C_{\theta_0}$ is zero of $f(u)$, i.e.,

$$f(u_0) = (1 - u_0^2)(\alpha - \beta u_0) - (b - a u_0)^2 = 0 \quad \ldots (12)$$

— Moreover, we would not want any other value of u to be "allowed", i.e., $f(u) < 0$ for $u \neq u_0$ $(= \dot{u}^2)$

(In this way, $u = u_0$ for all time.)

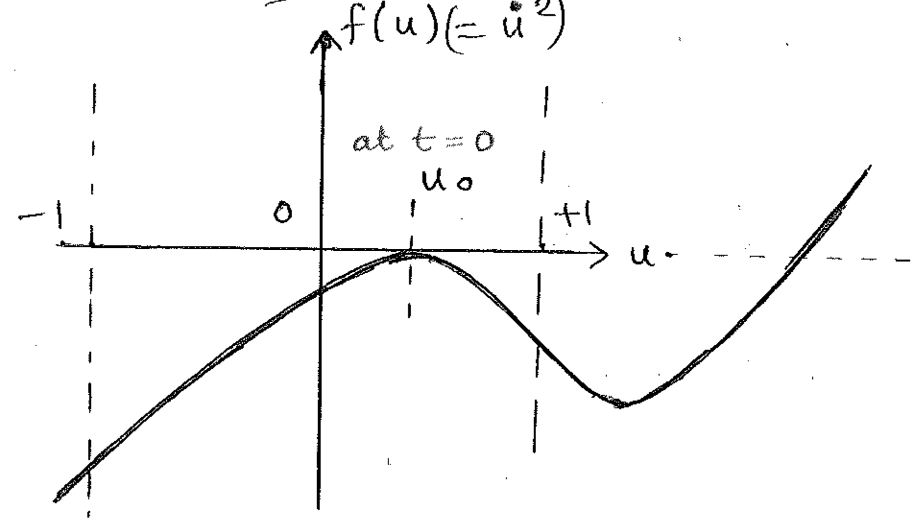
— From figure on page 6 of previous note, it is then clear that u_0 should be a double root of $f(u) = 0$ $(u_1 = u_2 = u_0)$: equivalently $f'(u_0) = 0$

[check: for cubic (or higher) polynomial $f(u)$, $f(u_0) = 0$ and $f'(u_0) = 0$ gives

$$f(u) = \underbrace{f(u_0)}_{=0} + \underbrace{f'(u_0)}_{=0}(u - u_0) + f''(u_0)(u - u_0)^2/2! + f'''(u_0)(u - u_0)^3/3!$$

$$= (u-u_0)^2 \left\{ f''(u_0)/2 + f'''(u_0)(u-u_0)/3! \right\}$$

i.e, u_0 is double root of $f(u) = 0$.]



uniform precession, without nutation

thus, we also have

$$f'(u_0) = 0 = -2u_0(\alpha - \beta u_0) - \beta(1-u_0^2) + 2a(b-au_0) \quad \cdots (13)$$

[This is sort of analogous to $\boxed{\text{circular}}$ orbit for central force corresponding to r_0 being extremum of $V'(r)$.] → $\boxed{\text{similar}}$ to relating E, l & r for $\boxed{\text{circular}}$ orbit

— Our goal is then the following: given $\boxed{\theta_0}$ and $\boxed{\omega_3}$ (constant spin of top), what's $\boxed{\dot{\phi}}$ needed for this motion? Eqs. (12) & (13) give

[using $\dot{\phi} = (b-au)/(1-u^2)$]

$$\tfrac{1}{2}\beta = a\dot{\phi} - \dot{\phi}^2 u_0,$$

i.e., using $I_1 a = I_3 \omega_3$ and $\beta = 2Mgl/I_1$, we have 2 possible values of $\boxed{\dot{\phi}}$ given by $\boxed{\text{quadratic}}$ equation

$$Mgl = \dot{\phi}(I_3\omega_3 - I_1\dot{\phi}\cos\theta_0) \quad \cdots (14)$$

[Check: $g \to 0$, i.e, $\boxed{\text{free}}$ top in Eq. (14) gives

$$\dot{\phi} = I_3\omega_3/(I_1\cos\theta_0) = \frac{L_3/\cos\theta_0}{I_1} = L/I_1$$ as before!]

— Also, existence of solutions for $\dot{\phi}$ requires discriminant > 0, i.e.,

$$(I_3\omega_3)^2 - 4(-I_1 c_{\theta_0})(-Mg\ell) > 0 \text{ or}$$

$$\boxed{\omega_3 > 2/I_3 \sqrt{Mg\ell I_1 c_{\theta_0}}} \quad \ldots (15)$$

i.e., for given θ_0, top has to be spinning $\boxed{\text{fast}}$ enough so as to allow uniform precession: for smaller ω_3, top will fall over (i.e., can't support uniform precession without nutation).

— $\boxed{\text{Sanity}}$ check: $\bar{\omega} = \underbrace{\dot{\psi}}_{\substack{\text{constant} \\ [=(\omega_3 - \dot{\phi} c_\theta)]}} \underbrace{\bar{e}_3}_{\substack{\text{moving} \\ \text{in space frame} \\ \text{since } \phi \neq \text{constant}}} + \cancel{\dot{\theta}}_{0}\, \bar{e}_1' + \underbrace{\dot{\phi}}_{\text{constant}} \underbrace{\tilde{\bar{e}}_3}_{\text{fixed (space) axis}}$ $\Rightarrow$ $\bar{\omega} \neq$ constant in space-frame (as expected from torque $\neq 0$ $\Rightarrow \bar{L} \neq$ constant)

$\boxed{(2)}$. $\boxed{\text{"Sleeping" top}}$

— Suppose we start with a top spinning in upright position, i.e., $\theta = 0$, $\dot{\theta} = 0$ @ $t = 0$ (so that $u = 0$, $\dot{u} = 0$), with $\omega_3 \neq 0$ (at least "blindly")

— Note that we can't really use $\dot{u}^2 = f(u)$ at $t = 0$ since that was derived assuming $(1-u^2) = S_\theta^2 \neq 0$

— However, at $t = 0$, we have $\bar{e}_3$ (body) $= \tilde{\bar{e}}_3$ (space) so that $\bar{L} \cdot \bar{e}_3 (= p_\psi) = \bar{L} \cdot \tilde{\bar{e}}_3 (= p_\phi)$, (see previous note)

i.e., $b(= p_\phi / I_1) = a\left(= \underbrace{I_3 \omega_3}_{p_\psi} / I_1\right)$

[p_ψ, ϕ of course stay the same constants of motion]

— Also, E @ $t = 0$ (but stays same) $= \frac{1}{2} I_3 \omega_3^2 + Mg\ell$

$(\theta = 0;\ \dot{\theta} = 0\ \&\ \dot{\phi} = 0)$

so that $E' = E - \frac{1}{2} I_3 \omega_3^2 = Mgl$, i.e.,

$\alpha\left(= 2E'/I_1\right) = \beta\left(= 2Mgl/I_1\right)$

— Using $a = b$ & $\alpha = \beta$ from above, we get (now for $\theta \neq 0$!) $f(u) \rightarrow (1-u^2)\,\alpha(1-u) - a^2(1-u)^2$

$$= (1-u)^2 \left[\alpha(1+u) - a^2\right]$$

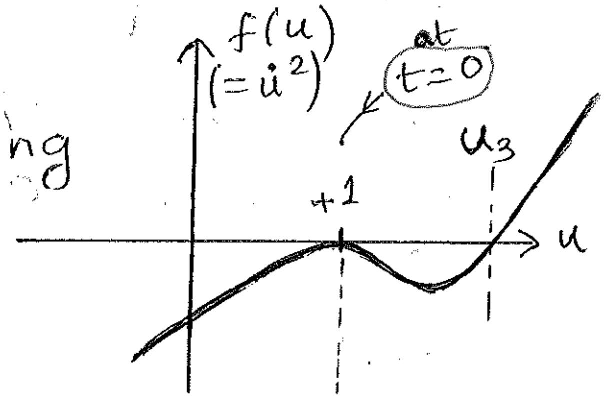
i.e., double root at $u = 1$ [so that $f'(1) = 0$]

and 3rd root is at $u_3 = \left(a^2/\alpha - 1\right)$

— Thus, we have $\boxed{2}$ possibilities:

(a) $\boxed{u_3 > 1}$ so that only physically allowed value is $u = 1$ [i.e., it's like case (1) above with $u_0 \rightarrow 1$]:

(stable) spinning upright



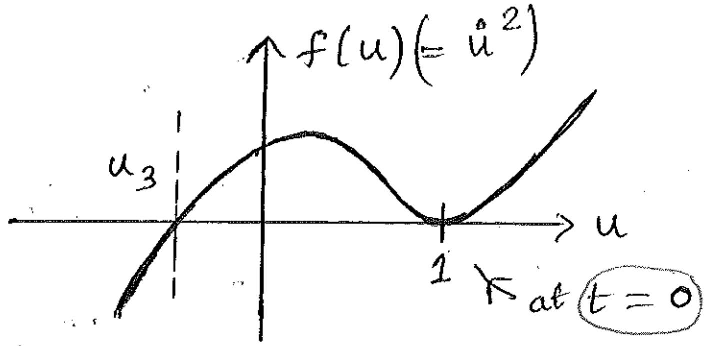
top is "sleeping"

— Thus, top continues to spin upright ($\boxed{\text{stable}}$ again position): this $\boxed{\text{requires}}$ $a^2/\alpha - 1 > 1$ (u_3 above > 1)

so that $\boxed{\omega_3^2 > 4 I_1 Mgl / I_3^2}$ corresponding to uniform precession case with $\boxed{\theta_0 = 0}$ (as expected).

(b) $\boxed{u_3 < 1}$ (or $\omega_3^2 < 4 I_1 Mgl/I_3^2$) so that other values of $u (< 1)$ are allowed [again, $u = 1$ is still double root or local minimum of $f(u)$]

Unstable spinning upright



— Thus, top is "allowed to" nutate between $\theta = 0$ ($u = 1$) and $\theta = \theta_3$ (corresponding to u_3)

— This looks like special case of "letting top go" (see $\boxed{\text{below}}$) : as θ increases ($u = \cos\theta$ reduces), $\dot{\phi}$ must "turn-on" etc.

— However, the difference is that in present case $\boxed{\theta = 0}$ can remain so [even if it is unstable], cf. below [θ (@ $t = 0$) $\neq 0$] where θ has to increase : that argument (see below) is based on non-zero torque, but for $\theta = 0$ (@ $t = 0$), there is $\boxed{\text{no}}$ torque (since gravity — always vertical! — is now through top-axis itself.), which is $\boxed{\text{why}}$ $\theta = 0$, $\dot{\theta} = 0$ remain that way [another argument is $\boxed{\text{below}}$]

since other θ's allowed

— In practice, even if we start with (i.e., $\theta = 0$ is stable) $\omega_3^2 > 4 I_1 M g l / I_3^2$, what will happen is that friction will reduce ω_3 below above critical value. Subsequently, any small disturbance (e.g., air resistance) will create instability, i.e., sleeping top "wakes up"!

In equations form, we have $\dot{u} = \sqrt{f(u)}$ so that $\ddot{u} = \frac{f'(u)}{2\sqrt{f}}\dot{u} = \frac{f'(u)}{2}$ [for $\dot{u} = 0$, just take limit suitably]. So, $\ddot{u}(t=0) = f'(1)/2 = 0$; similarly, higher time derivatives vanish $\Rightarrow \boxed{\theta = 0 \text{ for } t > 0}$ also.]

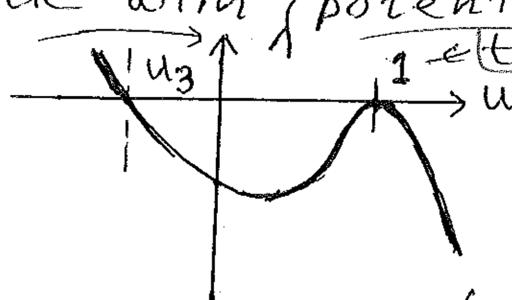
— In fact, above line of argument $\boxed{\text{explicitly}}$ shows why this case is unstable (cf. just saying earlier that other θ's or u's are "allowed" so that top will make an excursion there if perturbed).

— Namely, suppose some perturbation causes at an infinitesimal time after $t=0$ θ to deviate from 0, i.e., $\underset{\text{at } t=0}{u=1} \Rightarrow \underset{\text{at } t=0+\Delta t}{u=(1-\delta)}$ (where $\delta>0$ & $|\delta| \ll 1$). Then above relation, i.e., $\ddot{u} = f'(u)/2$ and looking at plot of $f(u)$, we get $\ddot{u}\,[\text{at } u = (1-\delta)] < 0$, i.e., $\dot{u}$ tends to decrease. But $\dot{u}(t=0 \text{ or } u=1)$ was 0. So, $\dot{u}(\text{at } u = 1-\delta) = -\epsilon$ ($\epsilon>0$, $|\epsilon| \ll 1$), i.e., $\boxed{u}$ tends to even more decrease at $t>0$ (or θ increases) so that top continues to fall, i.e., it is an unstable position.

— Equivalently, $\ddot{u} = f'(u)/2 = -\frac{d}{du}\left[\tfrac{1}{2}f(u)\right]$ suggests that the motion of top is "like" that of a 1D particle with (effective) potential energy $\left[-\frac{1}{2}f(u)\right]$.

$\left[-\frac{f}{2}(u)\right]$

(effective potential for u)



unstable upright top

— So, upright top ($u=1$) is (effectively) sitting on "top of a hill", i.e., highly susceptible to perturbation (even if it is in equilibrium, i.e., can stay put if not disturbed).

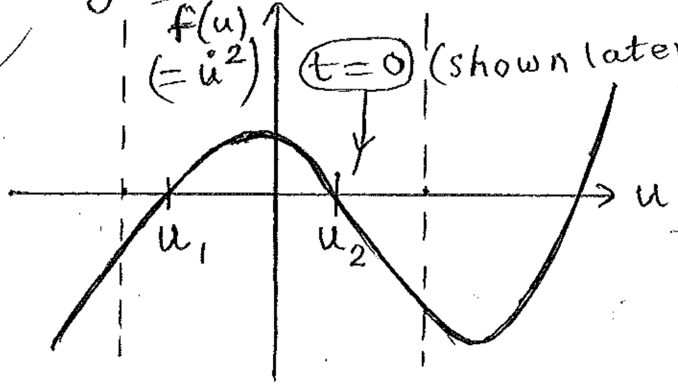
[Whereas, stable top of figure on page 4 is at bottom of hill] [Similarly, for figure on page 6 next.]

(3). $\boxed{\text{Letting the top go}}$ Suppose we spin the top ($\omega_3 \neq 0$) and "let it go" at some angle $\theta (\neq 0)$ at $\boxed{t=0}$, with no initial nutation or precession, i.e., $\boxed{\dot{\theta} = \dot{\phi} = 0 \text{ at } t=0}$. We expect top to fall due to gravity, i.e., θ will reduce: let's see using equations whether this happens.

— First of all, $\dot{u}(t=0) = -\underbrace{S_\theta(t=0)}_{\neq 0}\underbrace{\dot{\theta}(t=0)}_{0} = 0$

$\Rightarrow f[u(t=0)] = 0$, i.e., $u(t=0)$ is either $\boxed{u_1 \text{ or } u_2}$, one of $\boxed{2}$ turning points (see figure below). (zeroes of f)

Falling top, but bouncing back



(like general case on page 6 of previous note)

— Thus, other values of θ are allowed, where $\dot{\theta} = 0$, i.e., we (again) expect θ to not remain constant.

— However, as of now, we don't know whether $u(t=0)$ is at u_2 [i.e., $u(\text{at } t>0) < u(t=0)$ or θ increases] or at u_1 (θ decreases): again, gravitational expectation is it's former, but we need to show it via equations!

— First of all, $\dot{\phi}(t=0) = 0$ gives [see just below Eq. (11) of previous note] $b = a\underbrace{u(t=0)}_{\text{zero of } f}$, i.e.,

$\boxed{(b/a)}$ is zero of $f(u)$: let's check it.

— We have $f(b/a) = \left(1 - b^2/a^2\right)(\alpha - \beta\, b/a)$, since 2nd term in Eq. (11) for $f(u)$ vanishes for $u = b/a$.

- Now $\dot{\phi} = 0$ at $t = 0$ gives [using Eqs. (2) (4) & (5)] $b = a \cos[\theta(t=0)]$, i.e., $b = a\, u(t=0)$ [actually, same argument as before!] and (9), (10)

- On the other hand, Eqs. (3) with $\dot{\theta} = \dot{\phi} = 0$ at $t = 0$ gives $\alpha = \beta \cos[\theta(t=0)]$

- Combining above two relations shows
$\alpha/\beta = b/a$ so that indeed $f(b/a) = 0$

— Onto $\boxed{\text{showing}}$ that $\boxed{\dot{\theta}, \dot{\phi}}$ subsequently "turn on": let's do it by contradiction, i.e., suppose $\dot{\theta}, \dot{\phi}$ stay zero. ~~.~~

— However, then $\omega_{1,2} = \dot{\phi}\, s_\theta \, s_\psi, c_\psi \pm \dot{\theta}\, c_\psi, s_\psi$ remain zero, but then Euler's equations (with torque) along $\boxed{\bar{e}_{1,2}}$ cannot be satisfied: e.g., in general, we have along $\bar{e}_1$: $I_1 \dot{\omega}_1 + \omega_2 \omega_3 (I_3 - I_2) = \tau_1$ component of torque along $\bar{e}_1$

Now, LHS $= 0$ (again, $\omega_{1,2}$ stay 0), while RHS $= \underbrace{Mg\ell \sin\theta}_{\text{magnitude of torque}} \times \underbrace{\cos\psi}_{\text{component along } \bar{e}_1 \text{ (body x-axis)}}$

[Also, ψ is changing, i.e., $\neq \pi/2$ in general, since $\dot{\psi} = \omega_3 = \text{constant}$]

Thus, we have a contradiction so that our assumption that both $\dot{\theta}$ & $\dot{\phi}$ stay 0 is wrong $\Rightarrow$ one / both of $\dot{\theta}, \dot{\phi}$ must turn on at $t > 0$.

(see page 9 for shorter / different version)

[again, as expected from figure on page 6, i.e., values of θ where $\dot{\theta} \neq 0$ are allowed.] [see Eq. (3)]

— We also have $\left\{ E = \text{constant} = \underbrace{\frac{1}{2} I_3 \omega_3^2}_{\text{constant}} \right.$

$+ \frac{1}{2} I_1 \left(\dot{\theta}^2 + \dot{\phi}^2 s_\theta^2 \right) \Big\}$ = 0 at $t = 0$, but $>$ 0 for $t > 0$

$+ Mgl \cos\theta \Big\}$ must then reduce (as per above argument)

i.e., $\boxed{\theta \text{ must } \boxed{\text{increase}}}$ (again, $u \equiv \cos\theta$ then drops)

$\Rightarrow$ $u(t=0)$ must then be $\boxed{u_{2}}$ (i.e., larger root)

So, $\boxed{\text{top}}$ falls as expected!

— what about $\boxed{\dot{\phi}}$? We have $\dot{\phi} = \frac{b - au}{1 - u^2} = 0$ at $t = 0$

[again, $u(t=0) = b/a$ is (larger) root] : as u decreases, it is clear that $\dot{\phi}(t>0) \neq 0$, i.e., top starts to also precess! [Equivalently, Eq. (2) shows that to keep p_ϕ constant (with $I_3 \omega_3$ constant) as $\cos\theta$ drops, we have to turn on 1st term on RHS, i.e., $\dot{\phi}$]

— So, θ increases (u decreases) till we reach $\boxed{\text{other}}$ turning point, i.e., u_1; then top bounces back (in θ), i.e., motion in θ is bounded (again, like for central force)

— $\boxed{\text{Sanity}}$ check : at u_{1}, we have $\dot{\theta} = 0$ (just like at u_2, i.e., $t = 0$). Naively, intuitively, gravity should continue to make top fall (again, like we "guessed" initially for u_2), but we showed via equations above that top bounces back : this shows that "gravity makes top fall" might be too simple a reasoning!

— Then, what about repeating the argument (for θ to increase) using equations given for $u_{(2)}$ (at $t=0$), now doing it for $u_{\boxed{1}}$ at later time? However, $\mathring{\phi} \neq 0$ at u_1 (again, even if $\mathring{\theta} = 0$) — unlike at u_2 (at $t=0$) so that one can't really go through with the same argument

— x —

<u>Equivalent [to page (7)] version of argument</u> for either $\boxed{\mathring{\theta} \text{ or } \mathring{\phi}}$ to <u>turn on</u> for $\boxed{t > 0}$

— Again, assume $\mathring{\theta}, \mathring{\phi}$ stay 0 for all time, i.e., θ, ϕ remain at their initial values. So, $\overline{\omega} = \mathring{\psi}\, \underbrace{\overline{e}_3}_{\text{body}} + \mathring{\phi}\, \underbrace{\overline{e}_3}_{\text{space}} + \mathring{\theta}\, \underbrace{\overline{e}_1}_{\text{line of nodes}}$

$$\rightarrow \underbrace{\mathring{\psi}}_{\substack{\text{also constant} \\ (= \omega_3 \text{ for } \mathring{\phi} = 0)}} \underbrace{\overline{e}_3}_{\substack{\text{fixed even in space} \\ \text{frame (since so are} \\ \theta,\, \phi)}}$$

Thus, $\overline{L} = \overleftrightarrow{I} \cdot \overline{\omega} = I_3 \omega_3 \overline{e}_3$ is constant, but we do have non-zero torque here, giving a contradiction
$\Rightarrow$ one/both of $\mathring{\theta}, \mathring{\phi} \neq 0$ at $t > 0$ (as before)

— Using more of equations, we saw at bottom of page 5 that $\ddot{u} = f'(u)/2$. So, at $t = 0$, even though $\mathring{u} = 0$ (either at u_1 or u_2), we see that $\ddot{u}$ (at $t=0$) $\neq 0$; thus $\mathring{u} \neq 0$ at $t > 0$, i.e., $\mathring{\theta}$ is turned on (θ changes) [$\boxed{\text{c.f.}}$ sleeping top case on top of page 5, where $\ddot{u}$ at $t=0$ vanishes, i.e., $\boxed{f' = 0}$ (double root)]

$f'(u_{1,2}) \neq 0$

— x —

— $\boxed{\text{Finally}}$, we can make some <u>quantitative predictions</u> assuming "fast" top, i.e., rotational (initial) KE is much larger than maximum possible change in PE, i.e., $\frac{1}{2} I_3 \omega_3^2 \gg 2Mgl$

— In this case, we expect effect of torque (i.e., induced nutation & precession) to be sort of "perturbation" to rotation about its own axis; in particular extent of nutation $(u_2 - u_1)$ should decrease as ω_3 increases

— Let's turn to equations: we saw earlier (top of page 7) that $\alpha = \beta b/a$ and $u_2 (= u \text{ at } t=0) = b/a$. Then, after some algebra, we find that

$$f(u) = (u_2 - u)\underbrace{\left[(1-u^2)\beta - a^2(u_2-u)\right]}_{\text{contains other 2 roots of } f(u)=0}$$

— Let's look for a root in physically allowed region, i.e., the other turning point $u_1 < 1$:

$$(1-u_1^2) - \frac{a^2}{\beta}(u_2 - u_1) = 0$$

— Now, $a^2/\beta = \underbrace{\underbrace{I_3/I_1}_{O(1)} \underbrace{\frac{I_3 \omega_3^2}{2Mgl}}_{\gg 1 \text{ (for fast top)}}}_{} \gg 1$, while $(1-u_1^2) \lesssim O(1)$

So, we must have $(u_2 - u_1) \ll 1$ from root equation: as an approximation, we can then set $u_1 \approx u_2$ in 1st term, giving $(u_2 - u_1) \approx \dfrac{\beta \sin^2\theta_2}{a^2} \ll 1$

$$= \frac{I_1}{I_3}\frac{2Mgl}{I_3\omega_3^2}\sin^2\theta_2, \text{ i.e., } \frac{1}{\omega_3^2}$$

as anticipated, extent of nutation is small, reducing with ω_3.

— To finish the story, $\boxed{2^{nd}}$ root of above quadratic equation needs to be $\gg 1$ (i.e, unphysical): indeed, it is easy to see that it is given (approximately) by $\left(-\frac{a^2}{\beta}\right) \gg 1$, since then 1st term in root equation $\approx -u_1^2$ cancels $a^2 u_1/\beta$ from 2nd term (where we drop u_2) ... $\boxed{\text{more}}$ in GPS sec. 5.7