Here, we solve simple harmonic oscillator (SHO) using canonical transformation (CT), based on GPS section 9.3. The Lagrangian is

$$L(q,\dot{q}) = \frac{1}{2}m\dot{q}^2 - \frac{k}{2}q^2 \tag{1}$$

giving (as per usual procedure) the Hamiltonian:

$$\begin{aligned} H(q,p) &= \frac{p^2}{2m} + \frac{k}{2}q^2 \\ &= \frac{1}{2m}\left(p^2 + m^2\omega^2 q^2\right) \end{aligned} \tag{2}$$

where we write 2nd line in terms of $\omega \equiv \sqrt{k/m}$, anticipating (from usual method for solving) that this will be the frequency of oscillation of q. The above form of H, i.e., a sum of two squares, suggests that the following form of CT [where f is a (as yet) unknown function of P]to new coordinate Q and momentum P:

$$\begin{aligned} q(Q,P) &= \frac{f(P)}{m\omega}\sin Q \\ p(Q,P) &= f(P)\cos Q \end{aligned} \tag{3}$$

would simplify H, thus equations of motion (EOM). Such a change of variables is (roughly) analogous to going from Cartesian coordinates to polar. Indeed, we can easily find the transformed H, i.e., expressed in terms of the new variables (Q and P):

$$\tilde{H}(Q,P) = \frac{f^2(P)}{2m} \tag{4}$$

i.e., Q is cyclic so that P is constant of motion.

Of course, we still need to determine $f(P)$ such that the above is indeed a CT: we can be assured of this feature using Poisson brackets, i.e., we require:

$$\{q(Q,P), p(Q,P)\} = \frac{\partial q}{\partial Q}\frac{\partial p}{\partial P} - \frac{\partial q}{\partial P}\frac{\partial p}{\partial Q} = 1 \tag{5}$$

Plugging in Eqs. (3) into above, we straightforwardly (well, after some algebra!) find

$$f(P)\frac{df(P)}{dP} = m\omega \tag{6}$$

which is solved by

$$\frac{f^2(P)}{2} = m\omega P + c \tag{7}$$

where c is an arbitrary constant. Hence, we have

$$\tilde{H}(Q,P) = \omega P + c \tag{8}$$

Since a constant term in the Hamiltonian is irrelevant (i.e., does not affect the EOM), we can simply drop it. Equating above to the "original" H in Eq. (2), we get

$$P(q,p) = \frac{1}{2m\omega}\left(p^2 + m^2\omega^2 q^2\right) \tag{9}$$

Also, Eqs. (3) gives

$$Q(q,p) = \arctan\left(m\omega\frac{q}{p}\right) \tag{10}$$

As already mentioned, 2nd of Hamilton's equations [using Eq. (8)] gives

$$\begin{aligned} \dot{P} &= -\frac{\partial \tilde{H}}{\partial Q} \\ &= 0 \end{aligned} \tag{11}$$

i.e., P is a constant, which is related to the value of H (another constant, given that H is time-*in*dependent), i.e., energy (E):

$$P = \frac{E}{\omega} \tag{12}$$

Whereas, 1st Hamilton's equation:

$$\begin{aligned} \dot{Q} &= \frac{\partial \tilde{H}}{\partial P} \\ &= \omega \end{aligned} \tag{13}$$

simply (cf. solving *2nd* order differential equation for q, albeit a well-known one!) gives

$$Q = \omega t + \alpha \tag{14}$$

where α is a constant. Plugging in Eqs.(12) and (14) into Eq. (3) finally gives us the well-known solutions:

$$\begin{aligned} q &= \sqrt{\frac{2E}{m\omega^2}}\sin(\omega t + \alpha) \\ p &= \sqrt{2mE}\cos(\omega t + \alpha) \end{aligned} \tag{15}$$

i.e., oscillations with frequency ω and amplitude for q given by $\sqrt{2E/(m\omega^2)}$.