Here, we derive the Poisson brackets (PB's) of angular momentum components, i.e., $\{L_i, L_l\}$ using short-hand/compact notation (instead of doing it component-by-component, i.e., $\{L_1, L_2\} = L_3$, that was indicated in lecture and is in DT's Eq. 4.67). The *outline* of this procedure was given in lecture.

We begin with the definition

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} \tag{1}$$

which gives for the components

$$L_i = \epsilon_{ijk} x_j p_k \tag{2}$$

where ϵ is the totally antisymmetric Levi-Civita symbol, i.e., $\epsilon_{123} = 1 = \epsilon_{231} = -\epsilon_{132}$, while $\epsilon_{112} = 0$ etc. Here and henceforth, repeated indiced are summed. Similarly,

$$L_l = \epsilon_{lmn} x_m p_n \tag{3}$$

We will also make repeated use of the following relation

$$\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl} \tag{4}$$

We have from the definition of PB:

$$\begin{aligned}
\{L_i, L_l\} &= \frac{\partial L_i}{\partial x_r}\frac{\partial L_l}{\partial p_r} - \frac{\partial L_i}{\partial p_r}\frac{\partial L_l}{\partial x_r} \\
&= (\epsilon_{irk} p_k)(\epsilon_{lmr} x_m) - (\epsilon_{ijr} x_j)(\epsilon_{lrn} p_n) \qquad (5)
\end{aligned}$$

where 1st factor of 1st term comes from setting $j = r$ in L_i and "dropping" the x in it (due to that derivative), while 2nd factor in 1st term originates from setting $n = r$ in L_l and omitting the p from it. Similarly, for the 2 factors in the 2nd term.

Using the above indicated properties of ϵ, one then easily proceeed as follows:

$$\begin{aligned}
\{L_i, L_l\} &= \epsilon_{rki}\epsilon_{rlm} p_k x_m - \epsilon_{rij}\epsilon_{rnl} x_j p_n \\
&= (\delta_{kl}\delta_{im} - \delta_{km}\delta_{il}) p_k x_m - (\delta_{in}\delta_{jl} - \delta_{il}\delta_{jn}) x_j p_n \qquad (6) \\
&= p_l x_i - p_m x_m \delta_{il} - x_l p_i + x_n p_n \delta_{il} \\
&= x_i p_l - x_l p_i \\
&= (\delta_{ij}\delta_{lk} - \delta_{ik}\delta_{lj}) x_j p_k \\
&= \epsilon_{mil}\epsilon_{mjk} x_j p_k \qquad (7) \\
&= \epsilon_{ilm}(\epsilon_{mjk} x_j p_k) \\
&= \epsilon_{ilm} L_m \qquad (8)
\end{aligned}$$

In particular, Eq. (4) is used in getting to Eq. (6) from line before *and* its "inverse" in obtaining Eq. (7), while in the last line we re-use Eq. (2).