Here, we use the generating function of canonical transformation (CT) in order to derive the action-angle variables for the case of a time-independent 1D Hamiltonian (based on GPS section 10.6). The *general* idea behind applying CT (in particular, the action-angle variables) to solve problems is to go to new coordinate (Q) which is cyclic:

$$\tilde{H}(Q,P) \equiv H\Big[q(Q,P),p(Q,P)\Big] \tag{1}$$
$$= \tilde{H}(\text{only } P) \tag{2}$$

where P is the new momentum and q, p are old variables. Thus, we get (as usual)

$$\dot{P} = -\frac{\partial \tilde{H}}{\partial Q} \tag{3}$$
$$= 0 \tag{4}$$

so that P is constant, thus a function of energy E (which is also constant, given the time-independence of H). In turn,

$$\dot{Q} = \frac{\partial \tilde{H}}{\partial P} \tag{5}$$
$$= c(E) \tag{6}$$

where c is a (constant) function of E, i.e.,

$$Q = c(E)t + \text{constant} \tag{7}$$

Let us cast the above CT in terms of a generating function, $F_2(q,P)$, with (as usual)

$$p = \frac{\partial F_2(q,P)}{\partial q} \tag{8}$$

and

$$Q = \frac{\partial F_2(q,P)}{\partial P} \tag{9}$$

Plugging the 1st of above relations into $H(q,p) = E$ gives

$$H\left(q, \frac{\partial F_2\Big[q, P(E)\Big]}{\partial q}\right) = E \tag{10}$$

i.e., Eq. (10) for F_2 is "like" the Hamilton-Jacobi (H-J) equation for Hamilton's (explicitly time-*in*dependent) characteristic function $W(q,\alpha)$, where E was denoted by α in this context (for the 1D case), i.e.,

$$H\left(q, \frac{\partial W(q,\alpha)}{\partial q}\right) = \alpha \tag{11}$$

except that α in argument of W is replaced by a general *function* $P(\alpha)$ in going from Eq. (11) for W to Eq. (10) for F_2.

The simplest choice for P is in fact E (or α) itself, in which case above F_2 is *identical* to $W(q, \alpha)$; in this case, we get $c(E) = 1$ in Eqs.(6) and (7) so that

$$\begin{aligned} Q &= t + \text{constant} & (12)\\ &\neq \text{constant} & (13) \end{aligned}$$

i.e., $W(q, \alpha)$ generates a canonical transformation to a coordinate which is simply time (while new momentum is energy). On the other hand, recall that the main idea of the H-J method was to go to a *constant* new coordinate. However, note that in order to achieve that goal, we have to use Hamilton's *principal* (i.e., 'full" if you will) function as the (time-*dependent*) generating function of the CT [cf. characteristic function part only, i.e., $W(q, \alpha)$, used above]:

$$S(q, \alpha, t) = W(q, \alpha) - \alpha t \qquad (14)$$

so that indeed the transformed Hamiltonian vanishes:

$$\begin{aligned} K(Q, P, t) &= H(q, p) + \frac{\partial S}{\partial t} & (15)\\ &= \alpha - \alpha & (16) \end{aligned}$$

giving $\dot{Q} = 0$, i.e., $Q =$ constant.

Again, going back to Eq. (10) for F_2, we have the freedom (in general) to assign $P(E)$ to be a function of E instead of simply E. Now, suppose we have a bounded, periodic system with time period T (or angular frequency $\omega = 2\pi/T$). In this case, is it possible to *choose* $P(E)$ such that the corresponding new coordinate is the "angle", i.e., latter goes through 2π as the particle completes one cycle, i.e., over the period T? The answer is "Yes" (as we show next), the corresponding momentum and coordinate being denoted by $I(E)$ (called action) and θ.

We have change in new coordinate (*in general to begin with*, but still using the "final" notation of I, θ) over one cycle being given by

$$\begin{aligned} \Delta\theta &= \oint \frac{\partial \theta}{\partial q} dq & (17)\\ &= \oint \frac{\partial}{\partial q} \frac{\partial W(q, I)}{\partial I} dq, \text{ using } Eq.\ (9), \text{ with } Q \rightarrow \theta,\ P \rightarrow I \text{ and } F_2 \rightarrow W & (18) \end{aligned}$$

where we have used the *notation* W (i.e., that of Hamilton's characteristic function) for the generating function F_2, since (as mentioned above) it satisfies H-J-like equation. We can take the derivative with respect to I (which is a constant) outside the integral

$$\Delta\theta = \frac{d}{dI} \oint \frac{\partial W(q, I)}{\partial q} dq \qquad (19)$$

Here, one "worry/issue" in carrying out the differentiation with respect to I in Eq. (19), thus in going to simply Eq. (18) from it, is that endpoints of the motion also change with I

[in addition to the integrand, i.e., $\partial W(q, I)/\partial q$]. However, as we argued in the other/earlier way to derive the expression for $I(E)$, such effects are (in short/roughly speaking) higher order in the (infinitesimal) differentials, thus dropping-out when we take the limit to go to derivative. Finally, using Eq. (8), with $P \rightarrow I$ and $F_2 \rightarrow W$, we get

$$\Delta\theta = \frac{d}{dI} \oint p dq \tag{20}$$

So, requiring in our specific case

$$\Delta\theta = 2\pi \tag{21}$$

implies

$$I(E) = \frac{1}{2\pi} \oint p dq \tag{22}$$

i.e., same formula as we derived earlier (with*out* using generating function).