

Problems

1. Show, for the infinite well, that the average position $\langle x \rangle$ is independent of the quantum state.

Hint: Use the integral formula: $\int_0^{2n\pi} u \cos u \, du = 0$ (for integer n)

Answer: The way to solve this problem is by direct mathematical computation of the average position for the particle in the infinite well.

$$\langle x \rangle = \int_{-\infty}^{\infty} \psi_n^*(x) x \psi_n(x) \, dx$$

$$\text{Where } \psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$$

We do the integral making use of the trig identity, $2 \sin^2 \theta = 1 - \cos 2\theta$. Note that since the wave function is zero outside the box, we can replace the limits of integration with $[0, L]$.

$$\langle x \rangle = \frac{2}{L} \int_0^L x \sin^2\left(\frac{n\pi x}{L}\right) dx = \frac{1}{L} \int_0^L x - x \cos\left(\frac{2n\pi x}{L}\right) dx$$

$$\langle x \rangle = \frac{1}{L} \left[\frac{1}{2} x^2 \Big|_0^L - \int_0^L x \cos\left(\frac{2n\pi x}{L}\right) dx \right]$$

Making the substitution, $u \equiv \frac{2n\pi x}{L}$ and changing the limits of integration...

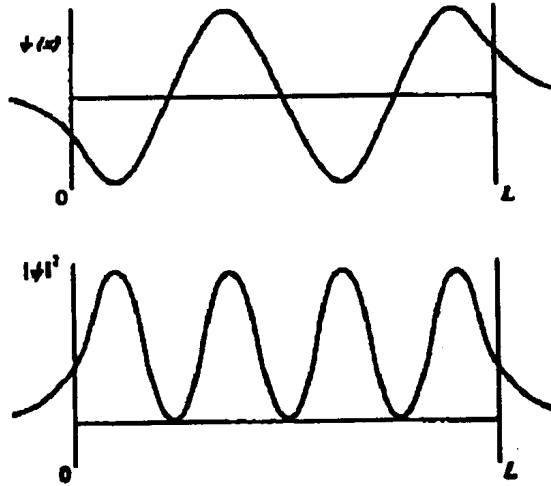
$$\langle x \rangle = \frac{L}{2} - \frac{L}{4n^2\pi^2} \int_0^{2n\pi} u \cos u \, du$$

$$\langle x \rangle = \frac{L}{2}$$

We find that this result is independent of the quantum state.

2. Carefully sketch the wave function and the probability density for the $n = 4$ state of a particle in a finite potential well.

Answer: Since the potential is symmetric, the 4th bound state wave function must have 3 nodes and display an odd symmetry about the midpoint of the well. Since the potential is constant inside the well, the wavelength and the amplitude of the sinusoidal curve are also constant.



3. **SMM, Chapter 5, problem 26.**

Hint: Check Example 5.13 if you are stuck.

Answer: At its limits of vibration $x = \pm A$, the classical oscillator has all its energy in potential form: $E = \frac{1}{2} m \omega^2 A^2$. If the energy is quantized according to

$E_n = (n + \frac{1}{2}) \hbar \omega$, then the corresponding amplitudes are

$$A_n = \sqrt{\frac{(2n+1)\hbar}{m\omega}}$$

4. Calculate $\langle x \rangle$, $\langle x^2 \rangle$, and Δx for a quantum oscillator in its ground state.

Hint 1: Is the integral, over all x , of an odd function zero?

Hint 2: Use the integral formula

$$\int_0^{\infty} u^2 e^{-au^2} du = \frac{1}{4a} \sqrt{\frac{\pi}{a}} \quad a > 0$$

Answer: For this problem we want to compute $\langle x \rangle = \int_{-\infty}^{\infty} \psi_n^(x) x \psi_n(x) dx$ and*

$\langle x^2 \rangle = \int_{-\infty}^{\infty} \psi_n^(x) x^2 \psi_n(x) dx$, for the quantum oscillator ground state (i.e.*

$\psi_0(x) = \left(\frac{m\omega}{\hbar\pi}\right)^{1/4} e^{-\left(\frac{m\omega}{2\hbar}\right)x^2}$). It is important to note that the ground state wave

function is a Gaussian, which has an even symmetry around the origin. [Ground state functions of symmetric potentials share this property].

And so...

$$\langle x \rangle = \sqrt{\frac{m\omega}{\hbar\pi}} \int_{-\infty}^{\infty} x e^{-\left(\frac{m\omega}{\hbar}\right)x^2} dx = 0$$

Since the integrand is made up of the multiplication of an even function (Gaussian) and an odd function (x), the resulting function is therefore odd. When we integrate such a function over all of x , the integral over the negative half-axis ($x < 0$) exactly cancels that over the positive half-axis ($x > 0$). Easy enough!

For the calculation of $\langle x^2 \rangle$, however, the integrand is even and the two half-axes contribute equally, giving

$$\begin{aligned} \langle x^2 \rangle &= \sqrt{\frac{m\omega}{\hbar\pi}} \int_{-\infty}^{\infty} x^2 e^{-\left(\frac{m\omega}{\hbar}\right)x^2} dx = 2 \sqrt{\frac{m\omega}{\hbar\pi}} \int_0^{\infty} x^2 e^{-\left(\frac{m\omega}{\hbar}\right)x^2} dx \\ \langle x^2 \rangle &= 2 \sqrt{\frac{m\omega}{\hbar\pi}} \frac{\hbar}{4m\omega} \sqrt{\frac{\hbar\pi}{m\omega}} = \frac{\hbar}{2m\omega} \end{aligned}$$

And so

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{\frac{\hbar}{2m\omega}}$$

5. SMM, Chapter 6, problem 1.

Answer: (a) The reflection coefficient is the ratio of the reflected to the incident wave intensity, or

$$R = \frac{\left| \frac{1}{2}(1-i) \right|^2}{\left| \frac{1}{2}(1+i) \right|^2} = \frac{(1-i)(1+i)}{(1+i)(1-i)} = 1$$

(b) To the left of the step, the wave function corresponds to a free particle and thus $k = \sqrt{\frac{2mE}{\hbar^2}}$. On the other hand, to the right of the step, wave function is a

decaying exponential where $k = \sqrt{\frac{2m(U-E)}{\hbar^2}}$. Setting these two equations equal to each other

$$\begin{aligned} \sqrt{\frac{2mE}{\hbar^2}} &= \sqrt{\frac{2m(U-E)}{\hbar^2}} \\ E &= U - E \\ \frac{E}{U} &= \frac{1}{2} \end{aligned}$$

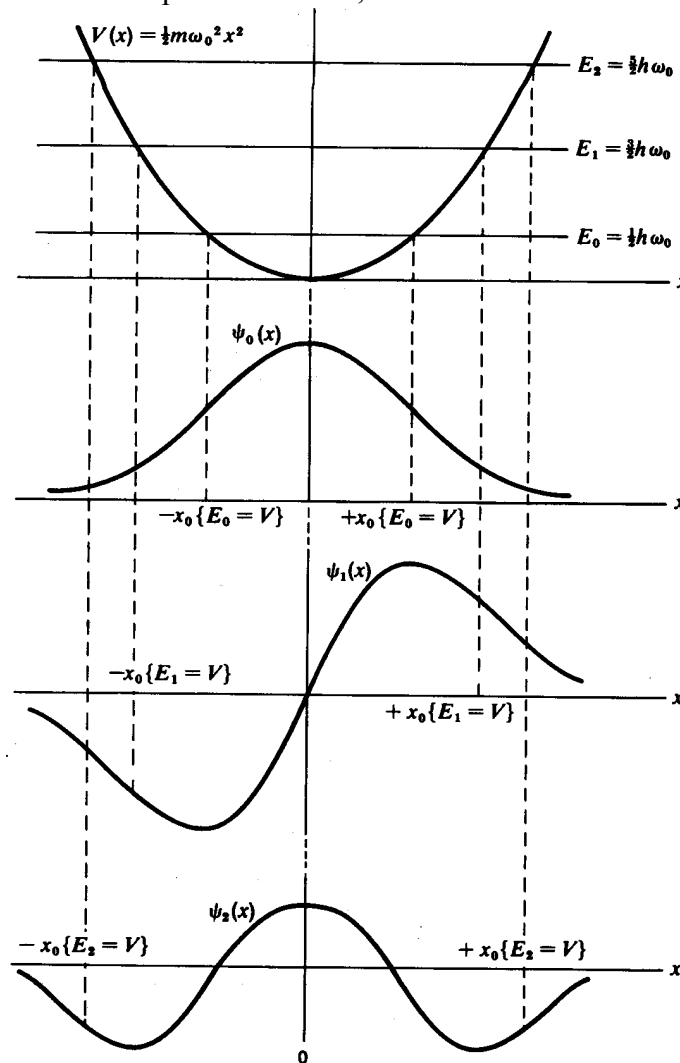
(c) The penetration depth is given by $\delta = \frac{1}{k} = \frac{\hbar}{\sqrt{2mE}}$. Letting $E = 10 \text{ MeV}$,

$m_{\text{proton}} = 938.3 \frac{\text{MeV}}{c^2}$ and using $\hbar = 197.3 \frac{\text{nm} \cdot \text{eV}}{c}$ (which comes from the fact that $hc = 1239.7 \text{ nm} \cdot \text{eV}$).

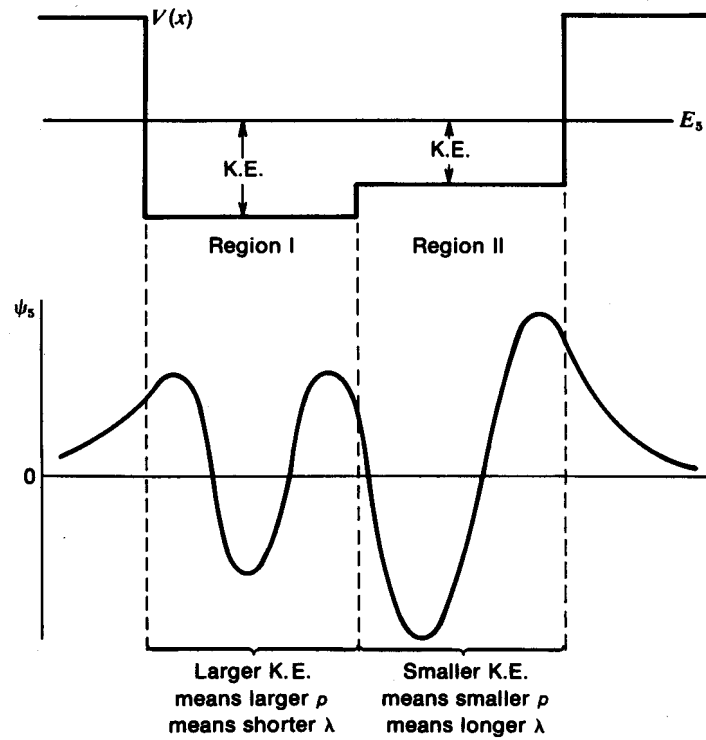
$$\delta = \frac{197.3 \text{ nm} \cdot \text{eV}/c}{\sqrt{2(938.3 \times 10^6 \text{ eV}/c^2)(10 \times 10^6 \text{ eV})}} = 1.44 \times 10^{-6} \text{ nm} = 1.44 \text{ fm}$$

6. Sketch careful, qualitatively accurate plots for the stated wave functions in each of the potentials shown. Check that your wave function has the correct symmetry, number of nodes, relative wavelengths, maximum values of amplitudes and relative rate of decrease outside the well.

(a) The ground state, 1st and 2nd excited wave functions of the quantum oscillator. Realize that this corresponds to the 1st, 2nd and 3rd bound state.



(b) The 5th bound state of the finite square well with a two level floor.



(c) The 5th bound state of the finite square well with a ramped floor.

