**PHYS420 (Spring 2002)**
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**Homework # 10**
**(due Monday, April 29th , 2002)**

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## Problems

1. **Matter Waves**. Show that the group velocity ($v_{group}$) of matter waves is related to the phase velocity ($v_{phase}$) by the following expression: $v_{group} = \frac{c^2}{v_{phase}\big|_{k_0}}$.

*Answer: The general relation between the group and phase velocities is given by*

$$v_{group} = v_{phase}\big|_{k_0} + k\frac{dv_{phase}}{dk}\bigg|_{k_0}$$

*Recalling that for matter waves* $v_{phase} = c\sqrt{1+\left(\frac{mc}{\hbar k}\right)^2}$ *, then all we have to do is differentiate and simplify.*
*Differentiating with respect to k we find*

$$\frac{dv_{phase}}{dk} = -\frac{c}{k}\frac{\left(\frac{mc}{\hbar k}\right)^2}{\sqrt{1+\left(\frac{mc}{\hbar k}\right)^2}}$$

*Substituting and simplifying*

$$v_{group} = c\sqrt{1+\left(\frac{mc}{\hbar k_0}\right)^2} - k_0\frac{c}{k_0}\frac{\left(\frac{mc}{\hbar k_0}\right)^2}{\sqrt{1+\left(\frac{mc}{\hbar k_0}\right)^2}}$$

$$v_{group} = \frac{c^2}{c\sqrt{1+\left(\frac{mc}{\hbar k_0}\right)^2}} = \frac{c^2}{v_{phase}\big|_{k_0}}.$$

2. Show that a monochromatic plane wave wavefunction $\Psi(x,t) = Ae^{i(kx-wt)}$ satisfies the *time-dependent* Schrödinger equation, $-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + U(x)\Psi = i\hbar\frac{\partial\Psi}{\partial t}$, assuming that $U(x) = 0$.

*We differentiate the wavefunction to find all the terms*

$$\frac{\partial \psi}{\partial x} = ikAe^{i(kx-wt)}, \ \frac{\partial^2 \psi}{\partial x^2} = -k^2 Ae^{i(kx-wt)}, \ \frac{\partial \psi}{\partial t} = -iwAe^{i(kx-wt)}$$

*And we simply plug into the Schrödinger equation with* $U(x) = 0$

$$-\frac{\hbar^2}{2m}(-k^2 Ae^{i(kx-wt)}) = i\hbar(-iwAe^{i(kx-wt)})$$

$$\frac{\hbar^2 k^2}{2m} = \hbar w$$

$$E = E$$

*The equation is satisfied.*

3. **SMM, Chapter 5, problem 1.**

*Answer: For the function to be a Schrödinger wavefunction, it must be singled-valued, continuous and finite everywhere.*

*(a) No. It diverges as* $x \to \infty$. $\psi(x)$ *must be finite everywhere.*

*(b) Yes.*

*(c) Yes.*

*(d) No.* $\psi(x)$ *is not singled valued everywhere.*

*(e) No.* $\psi(x)$ *is not continuous everywhere.*

4. **SMM, Chapter 5, problem 2.**

*Answer: (a) Normalization requires that* $\int_{-\infty}^{\infty} |\psi|^2 \, dx = 1$.

*Using our function and making use of the trig identity* $2\cos^2\theta = 1 + \cos 2\theta$.

$$\int_{-L/4}^{L/4} A^2 \cos^2\left(\frac{2\pi x}{L}\right) dx = 1$$

$$\frac{A^2}{2}\int_{-L/4}^{L/4} \left[1 + \cos\left(\frac{4\pi x}{L}\right)\right] dx = 1$$

$$\left| x + \frac{L}{4\pi}\sin\left(\frac{4\pi x}{L}\right)\right|_{-L/4}^{L/4} = \frac{2}{A^2}$$

$$\frac{L}{2} = \frac{2}{A^2} \quad \to \quad A = \frac{2}{\sqrt{L}}$$

*(b) The probability is just given by*

$$P_{[0,L/8]} = \int_0^{L/8} |\psi|^2 dx = \frac{A^2}{2}\left| x + \frac{L}{4\pi}\sin\left(\frac{4\pi x}{L}\right)\right|_0^{L/8} = \frac{2}{L}\left[\frac{L}{8} + \frac{L}{4\pi}\sin\left(\frac{\pi}{2}\right)\right]$$

$$P_{[0,L/8]} = \frac{1}{4} + \frac{1}{2\pi} = 0.409$$

5. **SMM, Chapter 5, problem 5.**

*Answer: (a) Since we know that the particle has zero energy, then our time-independent Schrödinger equation will look like this.*

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + U(x)\Psi = 0$$

*Since we know the wavefunction also, we are in a position to find the unknown potential. Taking the appropriate derivatives…*

$$\frac{\partial\psi}{\partial x} = A\left(1 - \frac{2x^2}{L^2}\right)e^{-\frac{x^2}{L^2}}$$

$$\frac{\partial^2\psi}{\partial x^2} = A\left(\frac{4x^3}{L^4} - \frac{6x}{L^2}\right)e^{-\frac{x^2}{L^2}}$$

*And plugging in…*

$$-\frac{\hbar^2}{2m}\left[A\left(\frac{4x^3}{L^4} - \frac{6x}{L^2}\right)e^{-\frac{x^2}{L^2}}\right] + U(x)\left[Ax\, e^{-\frac{x^2}{L^2}}\right] = 0$$

*After some manipulation we find…*

$$U(x) = \frac{\hbar^2}{2mL^2}\left[\frac{4x^2}{L^2} - 6\right]$$

*(b) This is the equation of a parabola that is centered at* $\left[0, -\frac{3\hbar^2}{mL^2}\right]$*, concave up since* $\psi''(x) = \frac{4\hbar^2}{mL^4} > 0$*, and crosses the x-axis at* $x = -\sqrt{\frac{3}{2}}\,L$ *and* $\sqrt{\frac{3}{2}}\,L$.

6. **SMM, Chapter 5, problem 9.**

*Answer: If we treat the nuclear potential as an infinite square well of length L =* $10^{-5}$ *nm, then the energies are given by* $E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}$*. The change in energy of a proton (*$m_p = 1.673x10^{-27}\,kg = 938.3\frac{MeV}{c^2}$*) in a transition from the first excited state (n=2) to the ground state (n=1) is then given by*

$$\Delta E = E_2 - E_1 = \frac{3h^2}{8m_p L^2} = \frac{3}{8}\frac{\left(1239.7\, eV\cdot nm/c\right)^2}{\left(10^{-5}\, nm\right)^2 938.3x10^6\, eV/c^2} = \textbf{6.14 MeV}$$

*The wavelength is just*

$$\lambda = \frac{hc}{\Delta E} = \frac{1239.7eV\cdot nm}{6.14x10^6 eV} = \textbf{2.02x10}^{\textbf{-4}}\textbf{ nm [gamma rays]}$$

7. **SMM, Chapter 5, problem 16.**

*Answer: In general, the probability of finding the particle within the interval [a,b] is given by* $P = \int_a^b |\psi(x)|^2\, dx$. *The normalized wave function of a particle in an infinitely deep potential is given by:* $\psi_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)$. *In this case, we can work out the value of this integral for all wave functions for an arbitrary interval [a,b].*

$$P = \int_a^b |\psi(x)|^2\, dx = \frac{2}{L}\int_a^b \sin^2\left(\frac{n\pi x}{L}\right)dx$$

$$P = \frac{2}{L}\int_a^b \frac{1}{2}\left[1-\cos\left(\frac{2n\pi x}{L}\right)\right]dx$$

$$P = \frac{2}{L}\left|\frac{x}{2} - \frac{L}{4n\pi}\sin\left(\frac{2n\pi x}{L}\right)\right|_a^b$$

*And thus…*

$$P_{[a,b]} = \frac{b-a}{L} + \frac{1}{2n\pi}\left[\sin\left(\frac{2n\pi a}{L}\right) - \sin\left(\frac{2n\pi b}{L}\right)\right]$$

*(a) To get the probability of the ground state electron being within .100 nm of the left hand wall of a .300 nm box (1/3 of the way) we let n = 1, L = 0.300 nm, a = 0.000 nm and b = 0.100 nm. More simply, you can leave the L as a variable and find the probability from 0 to L/3 (for n = 1). Of course, the answers are the same.*

$$P_{[0,\frac{L}{3}]} = \frac{1}{3} - \frac{1}{2n\pi}\sin\left(\frac{2n\pi}{3}\right) = \textbf{0.196} \text{ for } (n = 1).$$

*(b) For n = 100, we get*

$$P_{[0,\frac{L}{3}]} = \frac{1}{3} - \frac{1}{2(100)\pi}\sin\left(\frac{2(100)\pi}{3}\right) = \textbf{0.332} \text{ for } (n=100).$$

*(c) Yes. As n gets bigger we approach the classical value of 1/3.*