PHYS420 (Spring 2002)
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Homework # 5 Solutions

Problems

1. **SMM, Chapter 2, Problem 2.**

Assume that your skin can be considered a blackbody (which is probably not a very good assumption). One can then use Wien's displacement law, $\lambda_{\max}(m)T(K) = 2.898x10^{-3}\, m \cdot K$ *, to find the wavelength of the peak emission. Letting T = 35 Celsius = 308 K,*

$$\lambda_{\max} = 2.898x10^{-3}\, m \cdot K / 308K = 9.41x10^{-6}\, m = \boldsymbol{9409\ nm}$$

which is in the far infrared (heat).

2. **SMM, Chapter 2, Problem 3.**

According to classical mechanics, for a simple harmonic oscillator having amplitude A and spring constant k…

$$E_{total} = \frac{1}{2}kA^2,\ f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

(a) $E_{total} = \frac{1}{2}(25N/m)(0.4m)^2 = \boldsymbol{2.0\ J}$, *and* $f = \frac{1}{2\pi}\sqrt{\frac{(25N/m)}{(2kg)}} = \boldsymbol{0.56\ Hz}$.

(b) If the energy is quantized, it will be given by $E_n = nhf$ *. We find the value of the quantum number n simply by,*

$$n = \frac{E_n}{hf} = \frac{(2.0J)}{(6.63x10^{-34}\, J \cdot s)(0.56Hz)} = \boldsymbol{5.4x10^{33}}.$$

(c) At these frequencies, the energy carried away by one quantum change will be,

$$E = hf = (6.63x10^{-34}\, J \cdot s)(0.56Hz) = \boldsymbol{3.7x10^{-34}\ J}.$$

3. **SMM, Chapter 2, Problem 4.**

(a) Stefan's law tells us that the power per unit area emitted at all frequencies by a blackbody is proportional to the fourth power of its absolute temperature. Thus,

$$e_{total} = \sigma T^4 = (5.67x10^{-8}\, Wm^{-2}K^{-4})(3000K)^4 = \mathbf{4.59x10^6\ W/m^2}.$$

(b) If a light bulb is rated for P = 75 W, then…

$$e_{total} = \frac{Power}{Area} \quad \rightarrow Area = \frac{Power}{e_{total}} = \frac{(75W)}{(4.59x10^6\, W/m^2)} = \mathbf{16.3\ mm^2}.$$

4. Calculate the energy of a photon whose frequency is (a) $5x10^{14}$ Hz, (b) 10 GHz, (c) 30 MHz. Express your answers in electron volts. Also determine the

corresponding wavelengths for each case and what part of the EM spectrum this is. (This is essentially SMM problem 2.7 & 2.8 combined).

In all cases we use $E_n = nhf$ *and* $\lambda = \frac{c}{f}$ *...*

(a) E = (6.63x10^{-34} J.s)(5x10^{14} Hz) = 3.315x10^{-19} J = ***2.07 eV***
λ = (3x10^{8} m/s)/(5x10^{14} Hz) = 6x10^{-7} m = ***600 nm, visible (yellow)***

(b) E = (6.63x10^{-34} J.s)(10x10^{9} Hz) = 6.63x10^{-24} J = ***4.14 x 10^{-5} eV***
λ = (3x10^{8} m/s)/(10x10^{9} Hz) = ***0.03 m, microwave***

(c) E = (6.63x10^{-34} J.s)(30x10^{6} Hz) = 1.989x10^{-26} J = ***1.24 x 10^{-7} eV***
λ = (3x10^{8} m/s)/(30x10^{6} Hz) = ***10 m, radio***

5. **SMM, Chapter 2, Problem 11.**

In general, $Power = \frac{energy}{time} = \frac{nhf}{t} = (\frac{n}{t})hf$ *. Isolating for the number of photons per unit time…*

$$\frac{n}{t} = \frac{photons}{time} = \frac{Power}{hf} = \frac{P}{h}(\frac{\lambda}{c}) = \frac{(10W)(589.3x10^{-9}\,m)}{(6.63x10^{-34}\,J\cdot s)(3x10^{8}\,m/s)} = \mathbf{2.96 \times 10^{19}\ photons/s.}$$

6. *Extra Credit problem 1*. Using Planck's spectral distribution formula, *u(λ,T)*, and recalling that $e_{total} = \frac{c}{4}\int_0^\infty u(\lambda,T)d\lambda$, derive Stefan's law, $e_{total} = \sigma T^4$, for the total power per unit area radiated at all wavelengths. Work out the numerical value for the constant *σ*. *Useful hint:* $\int_0^\infty \frac{x^3}{(e^x-1)}dx = \frac{\pi^4}{15}$.

This problem is worked out in complete detail on page 70 of SMM. The derivation goes something like this.

$$e_{total} = \frac{c}{4}\int_0^\infty u(\lambda,T)d\lambda = \frac{c}{4}\int_{\lambda=0}^{\lambda=\infty}\frac{8\pi hc}{\lambda^5(e^{hc/\lambda k_bT}-1)}d\lambda = \frac{2\pi k_b^4T^4}{h^3c^2}\int_{x=0}^{x=\infty}\frac{x^3}{(e^x-1)}dx,$$

where we make use of the change of variable $x \equiv hc/\lambda k_b T$*. Note: In making the change of variables, don't forget to substitute* $d\lambda = -\frac{hc}{x^2k_bT}dx$*, and the limits of integration. There's a little bit of algebra in doing this.*

Finally, using the value for the integral…

$$e_{total} = \frac{2\pi^5 k_b^4}{15c^2h^3}T^4 = \sigma T^4$$

Substituting for k_b, c, and h, we have

$$\sigma = \frac{2\pi^5(1.381x10^{-23}\,J/K)^4}{15(3x10^8\,m/s)^2(6.626x10^{-34}\,J\cdot s)^3} = 5.67x10^{-8}\,W\cdot m^{-2}\cdot K^{-4}$$

7. *Extra Credit problem 2*. Using Planck's spectral distribution formula, $u(\lambda,T)$,
(a) Derive Wein's displacement law, $\lambda_{\max}T = const$. Assume that the transcendental equation, $x = 5(1-e^{-x})$, has a non-trivial solution given by x_0. *This comment will be clear as you work through the problem.*
(b) Using a dimensionless value for the non-trivial solution of $x_0 = 4.96511423$, work out the value and units of the constant.

In principle, the procedure is simple. To find the peak wavelength of the spectral distribution formula, we take the derivative with respect to wavelength, set it to zero and solve for the wavelength. And so…

$$u(\lambda,T) = \frac{8\pi hc}{\lambda^5(e^{hc/\lambda k_bT}-1)}$$

Setting the derivative to zero,

$$\frac{\partial u}{\partial \lambda} = 8\pi hc\left[\left(\frac{hc}{k_bT}\right)\frac{1}{\lambda^7}e^{hc/\lambda k_bT}\left(\frac{1}{e^{hc/\lambda k_bT}-1}\right)^2 - \frac{5}{\lambda^6}\left(\frac{1}{e^{hc/\lambda k_bT}-1}\right)\right] = 0$$

Solving for λ_{max}… (Note: All the λ's in the equations are λ_{max} but for brevity, I'm just calling them λ.)

$$\left(\frac{hc}{k_bT}\right)\frac{1}{\lambda^7}e^{hc/\lambda k_bT}\left(\frac{1}{e^{hc/\lambda k_bT}-1}\right)^2 = \frac{5}{\lambda^6}\left(\frac{1}{e^{hc/\lambda k_bT}-1}\right)$$

$$\frac{1}{5}\left(\frac{hc}{\lambda k_bT}\right)e^{hc/\lambda k_bT} = e^{hc/\lambda k_bT}-1$$

$$\frac{hc}{\lambda k_bT} = 5(1-e^{-hc/\lambda k_bT})$$

Letting $x \equiv hc/\lambda k_bT$ *we have*

$$x = 5(1 - e^{-x})$$

If we want to solve for λ_{max} we must solve for x. This is a transcendental equation in x that will not have an analytical solution but a numerical one. For the time being, let's assume that this numerical solution is x_0. And so,

$$x = x_0$$

$$\frac{hc}{\lambda_{\max} k_b T} = x_0$$

$$\lambda_{\max} T = \frac{hc}{x_0 k_b} = const.$$

Assuming that x_0 = 4.96511423 (which is dimensionless), we can solve for the constant.

$$\lambda_{\max} T = \frac{(6.626x10^{-34}\, J \cdot s)(3x10^{8}\, m/s)}{(4.96511423)(1.381x10^{-23}\, J/K)} = 0.002899\, m \cdot K$$