PHYS420 (Spring 2002)
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Homework # 3 Solutions

Problems

1. **SMM, Chapter 1, Problem 6.**

According to time dilation, a clock that is moving at some velocity v, will run at a slower rate than a clock that is at rest according to $t = \gamma\, t'$*, where t' is the time of the clock at rest (also known as the* ***proper*** *time) and* $\gamma \equiv \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$ *is the standard Lorentz factor. In other words, every t' interval of the moving clock (as read by a clock moving in their reference frame) will seem to last* $\gamma\, t'$ *to the outside observer. Therefore, if we want a moving clock to run at half the rate of the clock at rest, then what we really want is for 1 second of the moving clock to seem to last 2 seconds to the outside observer; mathematically speaking,* $t = 2\,t'$*. This is the same as asking what velocity will make the Lorentz factor equal to 2. Therefore, solving* $\gamma \equiv \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$ *for v as a function of γ we get* $v = \frac{c}{\gamma}\sqrt{\gamma^2 - 1}$*. Letting γ = 2 we find* $v = \frac{\sqrt{3}}{2}c = 0.866025c$*.*

2. **SMM, Chapter 1, Problem 7.**

We follow a similar line of reasoning as in the above problem. According to length contraction, a ruler that is moving at some velocity v, will have its length reduced compared to a ruler that is at rest according to $L = \frac{1}{\gamma}L'$*, where L' is the length of the ruler at rest (****proper*** *length). Therefore, for the observed length of a meter stick to shrink by a factor of ½ to 0.5 m (i.e.* $L = \frac{1}{2}L'$*), we again want a velocity that will make the Lorentz factor equal to 2. This is the same problem as above. To repeat, we solve* $\gamma \equiv \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$ *for v to get* $v = \frac{c}{\gamma}\sqrt{\gamma^2 - 1}$*. Letting γ = 2 we find* $v = \frac{\sqrt{3}}{2}c = 0.866025c$*.*

3. Muons are elementary particles with a (proper) lifetime of 2.2 μs. They are produced with very high speeds in the upper atmosphere when cosmic rays (high-energy particles from space) collide with air molecules. Take the height L_0 of the atmosphere to be 100 km in the reference frame of the Earth. (a) What is the

minimum speed that enables the muons to survive the journey to the surface of the Earth? (b) In the reference frame of the muon, what is the apparent thickness of the Earth's atmosphere?

For a given muon velocity v, the observer on Earth thinks that the muon lifetime has dilated to $t = \gamma\, t_0$. *Notice that I am now calling proper time* t_0 *not t'; this is to be consistent with the problem's use of* L_0 *as the proper length of the atmosphere. So in the Earth's reference frame, the muon is seen to travel a distance* L_0 *in time t. Conversely, in the muon reference frame, it travels a distance L in time* t_0. *Even though the two reference frames will never agree on the distance traveled or the lifetime, they will always agree on their relative speed. Therefore we can always write* $v = \frac{L_0}{t} = \frac{L}{t_0}$. *Within the context of this problem (we are given* L_0*) let's use the first equality and substitute the non-proper time with* $t = \gamma\, t_0$.

$$v = \frac{L_0}{t} = \frac{L_0}{\gamma t_0} = \frac{L_0}{t_0}\sqrt{1-\frac{v^2}{c^2}}$$

Isolating v in the equation, $v = \frac{L_0}{t_0}\sqrt{1-\frac{v^2}{c^2}}$, *we find* $v = \frac{L_0}{t_0}\left(1+\frac{L_0^2}{t_0^2 c^2}\right)^{-1/2}$.

(a) Plugging $L_0 = 100$ *km and* $t_0 = 2.2$ *μs (changing units to meters and seconds, of course!), we get* $v = 0.999978c$.

(b) In the reference frame of the muon, the apparent thickness of the Earth's atmosphere is just $L = \frac{1}{\gamma} L_0 = L_0\sqrt{1-\frac{v^2}{c^2}}$. *Plugging v we find L = 659.99 m.*

4. The mean proper lifetime of π^+ mesons (other elementary particles) is 2.5×10^{-8} s. In a beam of π^+ mesons of speed 0.99c, what is the average distance a meson travels before it decays? What would this value be if the relativistic time dilation did not exist?

Due to time dilation, we see elementary particles travel a lot farther they we expect them to. In this problem we want to compute how far this is for a π^+ *mesons traveling at 0.99c. Notice that any length measurement is done from our point of view in the lab (by some poor underpaid overworked graduate student) so it will be a proper length. Using the above equation,* $v = \frac{L_0}{\gamma t_0}$, *we solve for* L_0.

$$L_0 = \gamma t_0 v = t_0 \frac{v}{\sqrt{1-\frac{v^2}{c^2}}}$$

(a) Plugging in for $v = 0.99c$ *and* $t_0 = 2.5 \times 10^{-8}$ *s, we find* $L_0 = 52.6$ *m.*

(b) If time dilation did not exist (the equivalent of $\gamma = 1$*) then* $L_0 = t_0\, v = 7.43$ *m.*

5. **SMM, Chapter 1, Problem 24.**

This problem is just a straight mapping of coordinates using the Lorentz transformation rules. In other words, we are given (x, y, z, t) and we want to compute (x', y', z', t'). For v = 0.7c, γ = 1.40. Thus,

$x_1' = \gamma\,(x_1 - v\,t_1) = 0$
$t_1' = \gamma\,(t_1 - (v\,x_1)/c^2) = 0$
$y_1' = y_1 = 0$
$z_1' = z_1 = 0$
$x_2' = \gamma\,(x_2 - v\,t_2) = (1.40)\,(100 - 0) = 140\ m$
$t_2' = \gamma\,(t_2 - (v\,x_2)/c^2) = (1.40)\,(0 - (0.7c*100)/c^2) = -0.33\ \mu s$
$y_2' = y_2 = 0$
$z_2' = z_2 = 0$

(a) Therefore,

$(x_1', y_1', z_1', t_1') = (0, 0, 0, 0)$
$(x_2', y_2', z_2', t_2') = (140\ m, 0, 0, -0.33\ \mu s)$

(b) $\Delta x' = x_1' - x_2' = 140\ m.$

Note: Don't be misled into thinking that this part violates length contraction. Remember that the length of an object is a spacetime measurement defined by different space but identical time coordinates. Similarly, the tick of a clock is defined by different time but identical space coordinates. In this problem, we are dealing with different space and time coordinates.

(c) Events are not simultaneous in S', event 2 occurs 0.33 μs earlier than event 1.

6. As a follow up to problem 3 from the previous homework assignment, show that the scalar wave equation, $\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+\frac{\partial^2\varphi}{\partial z^2}=\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2}$, is invariant under Lorentz transformations.

Same as homework 2 problem 3 but with the Lorentz transformation rules.

$$x' = \gamma(x - vt), \quad y' = y, \quad z' = z, \quad t' = \gamma(t - \frac{vx}{c^2})$$

The non-zero coefficients in the derivative expansion are…

$$\frac{\partial x'}{\partial x} = \gamma, \quad \frac{\partial t'}{\partial x} = -\gamma\frac{v}{c^2}, \quad \frac{\partial x'}{\partial t} = -\gamma v, \quad \frac{\partial t'}{\partial t} = \gamma, \quad \frac{\partial y'}{\partial y} = 1, \quad \frac{\partial z'}{\partial z} = 1$$

The derivatives now transform the following way…

$$\frac{\partial}{\partial x} = \gamma\left(\frac{\partial}{\partial x'} - \frac{v}{c^2}\frac{\partial}{\partial t'}\right) \quad , \quad \frac{\partial}{\partial y} = \frac{\partial}{\partial y'} \quad , \quad \frac{\partial}{\partial z} = \frac{\partial}{\partial z'} \quad , \quad \frac{\partial}{\partial t} = \gamma\left(\frac{\partial}{\partial t'} - v\frac{\partial}{\partial x'}\right)$$

Turning now to the scalar wave equation in the non moving coordinate frame,

$$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+\frac{\partial^2\varphi}{\partial z^2}=\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2}$$

Substituting the new derivatives

$$\gamma\frac{\partial}{\partial x}\left(\frac{\partial\varphi}{\partial x'} - \frac{v}{c^2}\frac{\partial\varphi}{\partial t'}\right) + \frac{\partial^2\varphi}{\partial y'^2} + \frac{\partial^2\varphi}{\partial z'^2} = \frac{\gamma}{c^2}\frac{\partial}{\partial t}\left(\frac{\partial\varphi}{\partial t'} - v\frac{\partial\varphi}{\partial x'}\right)$$

Distributing through, exchanging the order and substituting again,

$$\gamma^2\left[\frac{\partial}{\partial x'}\left(\frac{\partial\varphi}{\partial x'}-\frac{v}{c^2}\frac{\partial\varphi}{\partial t'}\right)-\frac{v}{c^2}\frac{\partial}{\partial t'}\left(\frac{\partial\varphi}{\partial x'}-\frac{v}{c^2}\frac{\partial\varphi}{\partial t'}\right)\right]+\frac{\partial^2\varphi}{\partial y'^2}+\frac{\partial^2\varphi}{\partial z'^2}=$$

$$\frac{\gamma^2}{c^2}\left[\frac{\partial}{\partial t'}\left(\frac{\partial\varphi}{\partial t'}-v\frac{\partial\varphi}{\partial x'}\right)-v\frac{\partial}{\partial x'}\left(\frac{\partial\varphi}{\partial t'}-v\frac{\partial\varphi}{\partial x'}\right)\right]$$

After distributing through and simplifying,

$$\gamma^2\frac{\partial^2\varphi}{\partial x'^2}-\frac{\gamma^2v^2}{c^2}\frac{\partial^2\varphi}{\partial x'^2}+\frac{\partial^2\varphi}{\partial y'^2}+\frac{\partial^2\varphi}{\partial z'^2}=\frac{\gamma^2}{c^2}\frac{\partial^2\varphi}{\partial t'^2}-\gamma^2\left(\frac{v}{c^2}\right)^2\frac{\partial^2\varphi}{\partial t'^2}$$

A little more manipulation yields…

$$\frac{\partial^2\varphi}{\partial x'^2}+\frac{\partial^2\varphi}{\partial y'^2}+\frac{\partial^2\varphi}{\partial z'^2}=\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t'^2}$$

You can see that this is the original scalar wave equation form. Therefore, it is invariant under Lorentz transformations.

7. Estimate the number of jellybeans needed to fill a one-liter jar.

As with any estimation question, there are multiple directions from which the problem can be approached. Here are two different approaches. Solution 1 illustrates a more algorithmic approach; solution 2 is more intuitive. In both solutions, it is understood that one-liter is equal to 1000 cubic centimeters (cc).

<u>Solution 1</u>

What is the approximate size a jellybean?

An examination of a jellybean reveals that is approximately the size of a small cylinder that measures about 2 cm long by about 1.5 cm in diameter.

Do jellybeans "completely fill the liter bottle"?

The irregular shape of jelly beans result in them not being tightly packet; approximately 80% of the volume of the bottle is filled.

The number of jellybeans is the occupied volume of the jar divided by the volume of a single jellybean: Number of beans = (Occupied Volume of Jar)/(Volume of 1 Bean)

The volume of one jellybean is approximated by the volume of a small cylinder 2 cm long and 1.5 cm in diameter,

Volume of 1 jellybean = $\pi h (d/2)^2 = 2\text{cm} \times 3\ (1.5\text{cm}/2)^2$

$= 27/8$ *cubic centimeters*

Thus the approximate number of beans in the jar is…

Number of beans = (.80 x 1000 cc)/(27/8 cc) = approx ***240 jellybeans.***

<u>Solution 2</u>

Visualize a paper cube that measures 1 cubic inch.

How many jellybeans will fit in the cube? Approximately 4

How many cubic inches are there in 1 liter? 1 inch = approx 2.54 centimeters.

Therefore 1 cubic inch = approx. 16 cubic centimeters (cc).

1000 cc / 16 cc = approx 62 cubic inches in one liter.

How many jelly beans are there in the one liter container?

62 x 4 = approximately ***248 jellybeans***