PHYS420 (Spring 2002)
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Homework # 2 Solutions

Problems

1. Generalize the Galilean transformation of coordinates to motion in three dimensions by showing that $\vec{r}' = \vec{r} - \vec{v}t$ & $t' = t$.

   In the derivation of the Galilean transformations that was done in class, we assumed that the motion of the moving frame was just in the positive x direction (i.e. $\vec{v} = v_x \hat{x}$). In this problem, we are going to generalize this to any direction (i.e. $\vec{v} = v_x \hat{x} + v_y \hat{y} + v_z \hat{z}$).

   Assuming that at time t = t' = 0 the two frames are together, then at some arbitrary time t later, the distance between the two frames will be $v_x t$ in the x direction (just like the class derivation), $v_y t$ in the y direction and $v_z t$ in the z direction. In vector notation, we write this as

$$\vec{d} = v_x t\ \hat{x} + v_y t\ \hat{y} + v_z t\ \hat{z} = (v_x \hat{x} + v_y \hat{y} + v_z \hat{z})\ t = \vec{v}t\,.$$

   For each direction, the connection between the coordinates in the moving frame (x', y', z', t') and the coordinates in the stationary frame (x, y, z, t) are just

$$x' = x - v_x\, t$$
$$y' = y - v_y\, t$$
$$z' = z - v_z\, t$$
$$t' = t$$

   Therefore, in vector notation,

$$\vec{r}\,' = x'\hat{x} + y'\hat{y} + z'\hat{z}$$
$$\vec{r}\,' = (x - v_x\, t)\hat{x} + (y - v_y\, t)\hat{y} + (z - v_z\, t)\hat{z}$$
$$\vec{r}\,' = (x\hat{x} + y\hat{y} + z\hat{z}) - (v_x \hat{x} + v_y \hat{y} + v_z \hat{z})\, t$$
$$\vec{r}\,' = \vec{r} - \vec{v}\, t$$

2. In a laboratory frame of reference, an observer notes that Newton's 2nd law is valid. (a) Show that it's also valid for an observer moving at constant speed relative to the laboratory frame *(we did this in class)* & (b) Show that it is not valid in a reference frame moving past with constant acceleration. *This problem is simply SMM Chapter 1, Problems 1 & 2 combined.*

In order to show that Newton's 2nd law (F=ma) is valid in a moving frame, we must look at how accelerations transform.

(a) For a frame moving at constant speed v (in the positive x direction) relative to a stationary frame, the Galilean coordinate transformations is just

$$x' = x - vt$$

Taking a time derivative (d/dt') and recalling that dt'=dt and that v is a constant, we find how the velocity transforms. This is simply the Galilean velocity addition law.

$$u' = \frac{dx'}{dt'} = \frac{d}{dt'}(x - vt) = \frac{dx}{dt'} - v\frac{dt}{dt'} = \frac{dx}{dt} - v\frac{dt}{dt} = u - v$$

To find the acceleration we take another time derivative (d/dt', with dt'=dt).

$$a' = \frac{du'}{dt'} = \frac{d}{dt'}(u - v) = \frac{du}{dt'} = \frac{du}{dt} = a$$

And so we find that the accelerations are identical. Therefore so are the forces,

$$F' = ma' = ma = F$$

(b) For a frame moving at constant acceleration a_0 (in the positive x direction) relative to a stationary frame we cannot use the standard Galilean transformation rules that we derived in class anymore. Assuming that at time t = t' = 0 the two frames are together, then at some arbitrary time t later, the distance between the two frames will be ½ a_0t^2 in the x direction. In this case, the new coordinate transformation is

$$x' = x - \frac{1}{2}a_0t^2$$

Taking time derivatives to find the velocity (and remembering that dt' = dt),

$$u' = \frac{dx'}{dt'} = \frac{d}{dt'}(x - \frac{1}{2}a_0t^2) = \frac{dx}{dt'} - a_0t\frac{dt}{dt'} = u - a_0t$$

Taking time derivatives to find the acceleration,

$$a' = \frac{du'}{dt'} = \frac{d}{dt'}(u - a_0t) = \frac{du}{dt'} - a_0\frac{dt}{dt'} = a - a_0$$

We see that the two accelerations are not identical. So in this case, Newton's 2nd law will not have the same value in the two different frames…

$$F' = ma' = m(a - a_0) = ma - ma_0 = F - ma_0\,!$$

3. What happens to Maxwell's equations under a Galilean transformation? In a stationary reference frame (K) in free space, the scalar field $\varphi(x, y, z, t)$ satisfies the scalar wave equation, $\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2}$. Show that the form of the wave equation is not invariant under Galilean transformations.

This question is really asking whether or not $\frac{\partial^2\varphi}{\partial x'^2} + \frac{\partial^2\varphi}{\partial y'^2} + \frac{\partial^2\varphi}{\partial z'^2} = \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t'^2}$ *is true under Galilean transformations. In general, we make use of the Chain rule to see how the derivatives transform. For the time being and to be as general as* *possible, let's not assume a specific form for the coordinate transformation rules. We'll assume that the coordinates in the moving frame are* (x', y', z', t') *and that they depend only on* (x, y, z, t). *Similarly, the coordinates in the stationary frame are* (x, y, z, t) *and they depend only on* (x', y', z', t'). *The derivate operators can therefore be expanded the following way…*

$$\frac{\partial}{\partial x} = \frac{\partial x'}{\partial x}\frac{\partial}{\partial x'} + \frac{\partial y'}{\partial x}\frac{\partial}{\partial y'} + \frac{\partial z'}{\partial x}\frac{\partial}{\partial z'} + \frac{\partial t'}{\partial x}\frac{\partial}{\partial t'}$$
$$\frac{\partial}{\partial y} = \frac{\partial x'}{\partial y}\frac{\partial}{\partial x'} + \frac{\partial y'}{\partial y}\frac{\partial}{\partial y'} + \frac{\partial z'}{\partial y}\frac{\partial}{\partial z'} + \frac{\partial t'}{\partial y}\frac{\partial}{\partial t'}$$
$$\frac{\partial}{\partial z} = \frac{\partial x'}{\partial z}\frac{\partial}{\partial x'} + \frac{\partial y'}{\partial z}\frac{\partial}{\partial y'} + \frac{\partial z'}{\partial z}\frac{\partial}{\partial z'} + \frac{\partial t'}{\partial z}\frac{\partial}{\partial t'}$$
$$\frac{\partial}{\partial t} = \frac{\partial x'}{\partial t}\frac{\partial}{\partial x'} + \frac{\partial y'}{\partial t}\frac{\partial}{\partial y'} + \frac{\partial z'}{\partial t}\frac{\partial}{\partial z'} + \frac{\partial t'}{\partial t}\frac{\partial}{\partial t'}$$

It looks painful, but it's straightforward. Now we can assume a set of transformation rules. In our example, we'll use the Galilean transformation rules.

$$x' = x - vt$$
$$y' = y$$
$$z' = z$$
$$t' = t$$

We can now compute the coefficients in the derivative expansion. This is the part that depends on your choice of transformation rules.

$$\frac{\partial x'}{\partial x} = 1 \quad , \quad \frac{\partial y'}{\partial x} = 0 \quad , \quad \frac{\partial z'}{\partial x} = 0 \quad , \quad \frac{\partial t'}{\partial x} = 0$$
$$\frac{\partial x'}{\partial y} = 0 \quad , \quad \frac{\partial y'}{\partial y} = 1 \quad , \quad \frac{\partial z'}{\partial y} = 0 \quad , \quad \frac{\partial t'}{\partial y} = 0$$
$$\frac{\partial x'}{\partial z} = 0 \quad , \quad \frac{\partial y'}{\partial z} = 0 \quad , \quad \frac{\partial z'}{\partial z} = 1 \quad , \quad \frac{\partial t'}{\partial z} = 0$$
$$\frac{\partial x'}{\partial t} = -v \quad , \quad \frac{\partial y'}{\partial t} = 0 \quad , \quad \frac{\partial z'}{\partial t} = 0 \quad , \quad \frac{\partial t'}{\partial t} = 1$$

So putting everything together, we see that the derivatives transform the following way:

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x'} \quad , \quad \frac{\partial}{\partial y} = \frac{\partial}{\partial y'} \quad , \quad \frac{\partial}{\partial z} = \frac{\partial}{\partial z'} \quad , \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial t'} - v\frac{\partial}{\partial x'}$$

Turning now to the scalar wave equation in the non moving coordinate frame,

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 \varphi}{\partial t^2}$$

Splitting the derivatives to be clear

$$\frac{\partial}{\partial x}\left(\frac{\partial \varphi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \varphi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial \varphi}{\partial z}\right) = \frac{1}{c^2}\frac{\partial}{\partial t}\left(\frac{\partial \varphi}{\partial t}\right)$$

Substituting the new derivatives

$$\frac{\partial}{\partial x}\left(\frac{\partial \varphi}{\partial x'}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \varphi}{\partial y'}\right) + \frac{\partial}{\partial z}\left(\frac{\partial \varphi}{\partial z'}\right) = \frac{1}{c^2}\frac{\partial}{\partial t}\left(\frac{\partial \varphi}{\partial t'} - v\frac{\partial \varphi}{\partial x'}\right)$$

Distributing through and exchanging the order of the derivatives,

$$\frac{\partial}{\partial x'}\left(\frac{\partial \varphi}{\partial x}\right) + \frac{\partial}{\partial y'}\left(\frac{\partial \varphi}{\partial y}\right) + \frac{\partial}{\partial z'}\left(\frac{\partial \varphi}{\partial z}\right) = \frac{1}{c^2}\left(\frac{\partial}{\partial t'}\frac{\partial \varphi}{\partial t} - v\frac{\partial}{\partial x'}\frac{\partial \varphi}{\partial t}\right)$$

Substituting again the new derivatives,

$$\frac{\partial}{\partial x'}\left(\frac{\partial \varphi}{\partial x'}\right) + \frac{\partial}{\partial y'}\left(\frac{\partial \varphi}{\partial y'}\right) + \frac{\partial}{\partial z'}\left(\frac{\partial \varphi}{\partial z'}\right) = \frac{1}{c^2}\left[\frac{\partial}{\partial t'}\left(\frac{\partial \varphi}{\partial t'} - v\frac{\partial \varphi}{\partial x'}\right) - v\frac{\partial}{\partial x'}\left(\frac{\partial \varphi}{\partial t'} - v\frac{\partial \varphi}{\partial x'}\right)\right]$$

After distributing through and simplifying we get…

$$\frac{\partial^2 \varphi}{\partial x'^2} + \frac{\partial^2 \varphi}{\partial y'^2} + \frac{\partial^2 \varphi}{\partial z'^2} = \frac{1}{c^2}\frac{\partial^2 \varphi}{\partial t'^2} - \frac{2v}{c^2}\frac{\partial}{\partial x'}\frac{\partial \varphi}{\partial t'} + \frac{v^2}{c^2}\frac{\partial^2 \varphi}{dx'^2}$$

You can see that this is not the original scalar wave equation form; we got two extra terms on the right side! Therefore, it is NOT invariant under Galilean transformations.

4. **SMM, Chapter 1, Problem 3.** A 2000-kg car moving with a speed of 20 m/s collides with and sticks to a 1500-kg car at rest at a stop sign. Show that because momentum is conserved in the rest frame, momentum is also conserved in a reference frame moving with a speed of 10 m/s in the direction of the moving car.

Let m_1 = 2000 kg, v_1 = +20 m/s, m_2 = 1500 kg & v_2 = 0 m/s. According to conservation of momentum in the stationary frame,

$$m_1 v_1 + m_2 v_2 = (m_1 + m_2) v_{final}$$

Solving for v_{final} and plugging the numbers,

$$v_{final} = \frac{m_1 v_1 + m_2 v_2}{(m_1 + m_2)} = +11.429...m/s$$

Now, let's look at the same situation from a reference frame moving with a speed of v = +10 m/s. In this new frame,

$$v_1' = v_1 - v = 20 - 10 = 10 \text{ m/s}$$

$v_2' = v_2 - v = 0 - 10 = -10\ m/s$

$v_{final}' = v_{final} - v = 11.429... - 10 = +1.429...\ m/s$

And so, we find that momentum in this new frame is conserved,

$$m_1 v_1' + m_2 v_2' = (m_1 + m_2) v_{final}'$$

$$(2000)(10) + (1500)(-10) = (3500)(1.429...)$$

$$5000 = 5000$$

5. *Michelson – Morley experiment.* Show that we were justified in keeping only the first term of the binomial expansion when deriving the *expected* fringe shift. If you recall, $Shift \cong \frac{2Lv^2}{\lambda c^2}$. In other words, calculate what the fringe shift would be if you kept the next term and compare it to the resolution of the experiment ($\sigma_{Fringe} = 0.01$ fringe). Are we justified?

The binomial expansion to the next term is $(1+x)^n \cong 1 + nx + \frac{1}{2}n(n-1)x^2$

In deriving the expected fringe shift, we had the following exact expression,

$$\Delta t = t_1 - t_2 = \frac{2L}{c}\left[\left(1 - \frac{v^2}{c^2}\right)^{-1} - \left(1 - \frac{v^2}{c^2}\right)^{-1/2}\right]$$

Approximating this using the binomial theorem to the next term we find,

$$\Delta t \cong \frac{2L}{c}\left[1 + \frac{v^2}{c^2} + \frac{v^4}{c^4} - 1 - \frac{1}{2}\frac{v^2}{c^2} - \frac{3}{8}\frac{v^4}{c^4}\right] = \frac{Lv^2}{c^3} + \frac{5}{4}\frac{Lv^4}{c^5}$$

The expected fringe is therefore,

$$Shift = \frac{2c\Delta t}{\lambda} \cong 2\frac{Lv^2}{\lambda c^2} + \frac{5}{2}\frac{Lv^4}{\lambda c^4} = 0.4400000055$$

Where we've plugged the numbers provided in the book (i.e. L = 11 m, v = 30 km/s, λ = 500 nm). Since this correction is much smaller than the resolution of the experiment, we are perfectly justified in keeping only the first term of the binomial expansion.

6. Synchronized clocks are stationed at regular intervals, 1million km apart, along a straight line. When the clock next to you reads 12 noon, what time do you *see* (assuming you have a really powerful telescope) on the 90th clock down the line?

If the clocks are spaced 1 million km apart, then the 90th clock down the line is 90 million km from the clock next to me (i.e. d = 9 x 10^{10} m). Light takes t = d/c = (9x10^{10} m)/(3x10^{8} m/s) = 300 seconds = 5 minutes to get to me. So the clock will always seem to be 5 minutes behind. In other words, when I see 12 noon on the clock by my side, the light that reaches me from the 90th clock down, must have left 5 minutes prior to that. Hence, the 90th clock down will read 11:55 am.

7. Solve the non-relativistic Newton's equation of motion ($\vec{F} = \frac{d\vec{p}}{dt}$) in the case of a constant force in the positive x direction ($\vec{F} = F\hat{x}$). As a boundary condition, let $x(t=0) = x_0$ and $x'(t=0) = v_{x0}$. *Ignore motion in the y & z directions.*

Ignoring motion in the y and z directions, we want to solve the one-dimensional 2^{nd} order differential equation for x(t).

$$F = \frac{dp}{dt} = \frac{d}{dt}(mv) = m\frac{d}{dt}\left(\frac{dx}{dt}\right) = m\frac{d^2x}{dt^2}$$

Let's initially leave it in terms of the velocity.

$$F = m\frac{dv}{dt}$$

This equation is separable and can be solved by integrating with respect to each variable subject to the initial conditions.

$$F\,dt = m\,dv$$

$$\int_0^t F\,dt = \int_{v_{x0}}^{v} m\,dv$$

$$Ft = mv(t) - mv_{x0}$$

Dividing by m to clean things up and replacing v with dx/dt,

$$\frac{F}{m}t = \frac{dx}{dt} - v_{x0}$$

Since this is again separable we repeat the procedure (subject to initial conditions),

$$\int_0^t \frac{F}{m}t\,dt = \int_{x_0}^{x} dx - \int_0^t v_{x0}dt$$

$$\frac{1}{2}\frac{F}{m}t^2 = x(t) - x_0 - v_{x0}t$$

Thus, we derive Newton's equation of motion (remember that F/m = a)…

$$x(t) = x_0 + v_{x0}t + \frac{1}{2}\frac{F}{m}t^2$$